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Modal analysis of periodically time-varying linear rotor systems

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Abstract

General rotor systems possess both stationary and rotating asymmetric properties, whose equation of motion is characterized by the presence of periodically time-varying parameters with the period of half the rotation. This paper takes two different approaches to develop the complex modal analysis method for periodically time-varying linear rotor systems: one approach by employing Floquet theory and another by coordinate transformation. The first approach, based on decomposition of state transition matrix, leads to the periodically time-varying eigensolutions, whereas the second approach transforms the finite order time-varying matrix equation into an equivalent infinite order time-invariant linear equation by introducing modulated coordinates, leading to an infinite set of constant eigensolutions. The relations between the eigensolutions obtained by two different approaches are derived and their features are compared.

1. Introduction

A rotor-bearing system consists of two parts: stator and rotor parts. The stator part includes the bearings, seals, housings and foundations, whereas the rotor part includes the rotating disks, shafts and blades. According to the mechanical properties of the rotor and stator parts, rotor systems may be classified into four types [1,2]: isotropic (symmetric) rotor system—both rotor and stator parts are axi-symmetric; anisotropic rotor system—the rotor part is axi-symmetric but the stator part is not; asymmetric rotor system—the stator part is axi-symmetric but the rotor is not; general rotor system—neither rotor nor stator parts are axi-symmetric. The general rotor system, an asymmetric rotor with anisotropic stator, thus reveals the coupled effects of anisotropic and asymmetric rotor systems.

The asymmetric (anisotropic) rotor system may look like a periodically time-varying linear system when the equation of motion is written in the stationary (rotating) coordinates, but it can be easily transformed into a time-invariant linear system by rewriting the equation of motion in the rotating (stationary) coordinates. Thus, the asymmetric (anisotropic) rotor system alone essentially reduces to a time-invariant linear system,

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whose modal analysis scheme is well established in the literature [1,2]. On the other hand, the general rotor system, which is inherently a periodically time-varying linear system, cannot be transformed into a time-invariant system by such a simple coordinate transformation. It leads to a difficulty in modal analysis of the general rotor system due to mathematical complexity associated with treatment of periodically time-varying system matrices.

Many attempts have been made for dynamic analysis of periodically time-varying linear systems by employing the periodically time-varying eigenvectors [3–6]. Since these methods were strictly based on the time domain analysis, the stability analysis was of major concern. For example, Sinha et al. [5] proposed use of the Liapunov–Floquet transformation for expanding the periodic system matrices in terms of the shifted Chebyshev polynomials of a same period, so that the original differential problem reduces to a set of linear algebraic equations. However, their method is still limited to enhancement of the stability analysis. Calico and Wiesel [7] applied the Floquet theory to develop a modal analysis method for periodically time-varying control systems, introducing the periodically time-varying eigenvectors derived from periodicity of the state transition matrix. Although their modal analysis method is mathematically sound, it requires the accurate integration of the state transition matrix over a period and it lacks the natural extension to the frequency domain analysis.

There are few investigations on complete modal analysis of periodically time-varying systems valid in both time and frequency domains. The major difficulty is due to the fact that the conventional modal analysis developed for linear time-invariant systems cannot be directly applicable to linear time-varying systems, unless they can be transformed into an equivalent time-invariant system [5,6]. Irretier [8] developed a mathematical foundation for modal testing of periodically time-varying rotor systems by expanding the periodically time-varying modal vectors in Fourier series and introducing an intuitive, but not rigorously proven, relation between modal parameters. In addition, although the resulting mathematical treatments are found to be correct, neither the computational procedure for eigensolutions nor the frequency domain analysis for modal testing was described.

For asymmetrical rotors with isotropic stators, the periodically time-varying linear differential equation expressed in the stationary coordinates can be transformed to the time-invariant linear differential equation expressed in the rotating coordinates or in the modulated stationary coordinates [9]. Then the modal analysis becomes essentially the same as the ordinary complex modal analysis method developed for anisotropic rotors, which possess asymmetric properties only in the stator part [10,11]. On the other hand, the asymmetric rotor system with anisotropic stator cannot be transformed to a finite order equation of motion with the time-invariant parameters by coordinate transformation only.

This paper introduces two different approaches, one using the Floquet theory and another using the (modulated) coordinate transformation, for complete complex modal analysis of a general rotor system, whose linear equation of motion is characterized by periodically time-varying parameters. Then the relations between two approaches are derived in order to clearly understand the eigenstructure of the system and the two methods are comparatively discussed in calculation of the eigensolutions and directional frequency response functions (dFRFs). Finally, a simple analysis model is treated in order to demonstrate the theoretical findings and the effectiveness of the coordinate transform method.

2. Complex modal analysis of periodically time-varying rotor systems

2.1. Equation of motion in complex form

For a rotor system with rotating and stationary asymmetry, the equation of motion can be conveniently written in the complex stationary coordinates, referring to Fig. 1, as [1,2,12,13]

$$\mathbf{M_f}\ddot{\mathbf{p}}(t) + \mathbf{C_f}\dot{\mathbf{p}}(t) + \mathbf{K_f}\mathbf{p}(t) + \{\mathbf{M_b}\ddot{\bar{\mathbf{p}}}(t) + \mathbf{C_b}\dot{\bar{\mathbf{p}}}(t) + \mathbf{K_b}\bar{\mathbf{p}}(t)\} + \mathrm{e}^{\mathrm{j}2\Omega t}\{\mathbf{M_r}\ddot{\bar{\mathbf{p}}}(t) + \mathbf{C_r}\dot{\bar{\mathbf{p}}}(t) + \mathbf{K_r}\bar{\mathbf{p}}(t)\} = \mathbf{g}(t). \quad (1)$$

Here, the $N \times 1$ complex response and force vectors, $\mathbf{p}(t)$ and $\mathbf{g}(t)$, defined by the real response vectors, $\mathbf{y}(t)$ and $\mathbf{z}(t)$, and the real excitation vectors, $\mathbf{f}_y(t)$ and $\mathbf{f}_z(t)$, respectively, are

$$\mathbf{p}(t) = \mathbf{y}(t) + j\mathbf{z}(t),\ \bar{\mathbf{p}}(t) = \mathbf{y}(t) - j\mathbf{z}(t),\ \mathbf{g}(t) = \mathbf{f}_y(t) + j\mathbf{f}_z(t),\ \bar{\mathbf{g}}(t) = \mathbf{f}_y(t) - j\mathbf{f}_z(t), \quad (2)$$

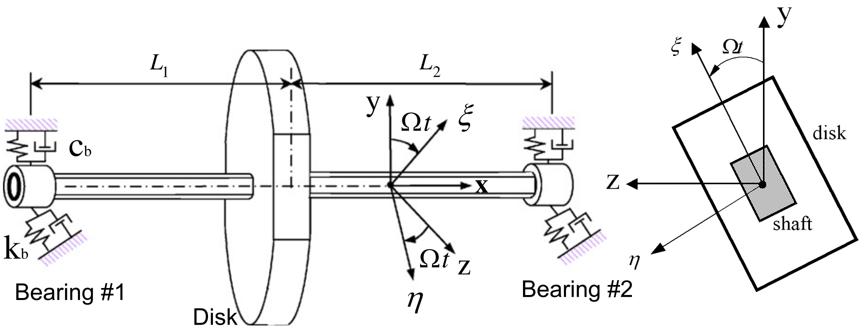

Fig. 1. General rotor system: simple analysis model.

where j means the imaginary number, N is the dimension of the complex coordinate vector, $\mathbf{g}(t)$ includes the force and moment; Ω is the rotational speed; '-' denotes the complex conjugate; $\mathbf{M}_i$, $\mathbf{C}_i$ and $\mathbf{K}_i$ denote the complex-valued $N \times N$ generalized mass, damping and stiffness matrices, respectively; and the subscripts $\mathbf{f}$, and, $\mathbf{b}$ and $\mathbf{r}$ refer to the mean value, and the deviatoric values for anisotropy (stationary asymmetry) and asymmetry (rotating asymmetry), respectively. For an isotropic rotor, $\mathbf{C_b} = \mathbf{K_b} = \mathbf{M_r} = \mathbf{C_r} = \mathbf{K_r} = \mathbf{0}$; for an anisotropic rotor, $\mathbf{M_r} = \mathbf{C_r} = \mathbf{K_r} = \mathbf{0}$; and, for an asymmetric rotor, $\mathbf{C_b} = \mathbf{K_b} = \mathbf{0}$. Note here that the periodically time-varying terms, which are preceded by $e^{j2\Omega t}$ in Eq. (1), inherently appear, as both rotating and stationary asymmetries exist in the system and that Eq. (1) includes the external and internal damping, gyroscopic moment and Coriolis effect. When either rotating or stationary asymmetry does not exist, the equation of motion becomes, or it can be transformed to, a time-invariant differential equation.

In the following sections, two different approaches, one using the Floquet theory and another using the coordinate transformation, are taken for complex modal analysis of the periodically time-varying linear rotor system (1). Then the relations between the modal solutions from both approaches are derived and their computational efficiency in calculating the eigensolutions and the dFRFs is discussed.

2.2. Complex modal solution by Floquet theory

(1) *Eigenvalue and adjoint problems*: From Eq. (1) and its complex conjugate form, the complex equation of motion can be constructed as

$$\mathbf{M}(t)\ddot{\mathbf{q}}(t) + \mathbf{C}(t)\dot{\mathbf{q}}(t) + \mathbf{K}(t)\mathbf{q}(t) = \mathbf{f}(t), \tag{3}$$

where

$$\mathbf{q}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \bar{\mathbf{p}}(t) \end{Bmatrix},\ \mathbf{f}(t) = \begin{Bmatrix} \mathbf{g}(t) \\ \bar{\mathbf{g}}(t) \end{Bmatrix},\ \mathbf{M}(t) = \begin{bmatrix} \mathbf{M_f} & \mathbf{M_r}e^{j2\Omega t} \\ \bar{\mathbf{M}}_\mathbf{r}e^{-j2\Omega t} & \bar{\mathbf{M}}_\mathbf{f} \end{bmatrix},$$

$$\mathbf{C}(t) = \begin{bmatrix} \mathbf{C_f} & \mathbf{C_b}+\mathbf{C_r}e^{j2\Omega t} \\ \bar{\mathbf{C}}_\mathbf{b} + \bar{\mathbf{C}}_\mathbf{r}e^{-j2\Omega t} & \bar{\mathbf{C}}_\mathbf{f} \end{bmatrix},\ \mathbf{K}(t) = \begin{bmatrix} \mathbf{K_f} & \mathbf{K_b}+\mathbf{K_r}e^{j2\Omega t} \\ \bar{\mathbf{K}}_\mathbf{b}+\bar{\mathbf{K}}_\mathbf{r}e^{-j2\Omega t} & \bar{\mathbf{K}}_\mathbf{f} \end{bmatrix}. \tag{4}$$

Eq. (3) can be rewritten in the state space form as

$$\mathbf{A}(t)\dot{\mathbf{w}}(t) = \mathbf{B}(t)\mathbf{w}(t) + \mathbf{F}(t), \tag{5}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{M}(t) \\ \mathbf{M}(t) & \mathbf{C}(t) \end{bmatrix},\ \mathbf{B}(t) = \begin{bmatrix} \mathbf{M}(t) & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}(t) \end{bmatrix},\ \mathbf{w}(t) = \begin{Bmatrix} \dot{\mathbf{q}}(t) \\ \mathbf{q}(t) \end{Bmatrix},\ \mathbf{F}(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{Bmatrix}. \tag{6}$$

Utilizing the Floquet theory for this periodically time-varying system in homogeneous part of Eq. (5) with the period $T = \pi/\Omega$, we can express the $4N \times 1$ complex state vector, $\mathbf{w}(t)$, in terms of the state transition matrix, $\Phi(t,\ t)$, as [3–7]

$$\mathbf{w}(t) = \Phi(t,0)\,\mathbf{w}(0), \tag{7}$$

where $\Phi(t,\ 0)$ satisfies the differential equation, subject to the initial condition $\Phi(0,0) = \mathbf{I}_{4N\times 4N}$,

$$\dot{\Phi}(t,0) = [\mathbf{A}^{-1}(t)\mathbf{B}(t)]\Phi(t,0) \tag{8}$$

and the matrix decomposition relation given, subject to $\mathbf{R}(0) = \mathbf{R}(T)$, by [4,7]

$$\Phi(t,0) = \mathbf{R}(t)\mathrm{e}^{\mathbf{J}t}\mathbf{R}^{-1}(0). \tag{9}$$

Here, $\mathbf{J}$ is the Jordan normal form of matrix, whose diagonal entries, $\mu_i,\ \ i = 1,2,\ldots,4N$, are termed Poincare exponents, equivalent to the eigenvalues for time-invariant systems. Note that, for the time-invariant system, $\Phi(t,0) = \mathrm{e}^{\mathbf{J}t}$, since it holds $\mathbf{R}(t) = \mathbf{R}(0) = \mathbf{R}$.
Substituting $t = T$ into Eq. (9), we obtain

$$\Phi(T,0) = \mathbf{R}(0)\mathrm{e}^{\mathbf{J}T}\mathbf{R}^{-1}(0) \tag{10}$$

or equivalently,

$$[\Phi(T,0) - \omega_i\mathbf{I}]\mathbf{R}(0) = \mathbf{0}. \tag{11}$$

It implies that $\omega_i = \mathrm{e}^{\mu_i \mathrm{T}}$ and $\mathbf{R}(0)$ are the eigenvalues (characteristic multipliers) and the corresponding matrix of eigenvectors, respectively, of the monodromy matrix $\Phi(T,0)$. Substituting Eq. (9) into Eq. (8), we obtain

$$\dot{\mathbf{R}}(t) = [\mathbf{A}^{-1}(t)\mathbf{B}(t)]\mathbf{R}(t) - \mathbf{R}(t)\mathbf{J} \tag{12a}$$

or equivalently,

$$\dot{\mathbf{r}}(t) = [\mathbf{A}^{-1}(t)\mathbf{B}(t) - \mu\mathbf{I}]\mathbf{r}(t), \tag{12b}$$

where $\mathbf{r}(t)$ is a column vector of $\mathbf{R}(t)$.
Now we can construct the adjoint problem, introducing the adjoint state vector $z(t)$, to the original system (5) with $\mathbf{F}(t) = \mathbf{0}$ as [4,7]

$$\dot{z}(t) = -[\bar{\mathbf{A}}^{-1}(t)\bar{\mathbf{B}}(t)]^{\mathrm{T}}z(t) \tag{13}$$

from which we can define the adjoint matrix $\bar{\mathbf{L}}(t)$ such that

$$\dot{\bar{\mathbf{L}}}(t) = -\left[\bar{\mathbf{A}}^{-1}(t)\bar{\mathbf{B}}(t)\right]^{\mathrm{T}}\bar{\mathbf{L}}(t) + \bar{\mathbf{L}}(t)\bar{\mathbf{J}} \tag{14a}$$

or

$$\dot{\mathbf{L}}^{\mathrm{T}}(t) = -\mathbf{L}^{\mathrm{T}}(t)[\mathbf{A}^{-1}(t)\mathbf{B}(t)] + \mathbf{J}\mathbf{L}^{\mathrm{T}}(t) \tag{14b}$$

or equivalently,

$$\dot{\bar{\mathbf{l}}}(t) = -[\{\bar{\mathbf{A}}^{-1}(t)\bar{\mathbf{B}}(t)\}^{\mathrm{T}} - \bar{\mu}\mathbf{I}]\bar{\mathbf{l}}(t) \tag{14c}$$

where $\bar{\mathbf{l}}(t)$ is a column vector of $\bar{\mathbf{L}}(t)$.
We can rewrite Eq. (12), using the identity relation $\mathbf{R}(t)\mathbf{R}^{-1}(t) = \mathbf{I}$, as

$$\dot{\mathbf{R}}^{-1}(t) = -\mathbf{R}^{-1}(t)[\mathbf{A}^{-1}(t)\mathbf{B}(t)] + \mathbf{J}\mathbf{R}^{-1}(t). \tag{15}$$

From direct comparison of Eq. (14b) with Eq. (15), we can obtain the bi-orthonormality conditions as [7]

$$\mathbf{L}^{\mathrm{T}}(t)\mathbf{R}(t) = \mathbf{I}_{4N\times 4N}, \tag{16a}$$

$$\dot{\mathbf{L}}^{\mathrm{T}}(t)\mathbf{R}(t) - \mathbf{L}^{\mathrm{T}}(t)[\mathbf{A}^{-1}(t)\mathbf{B}(t)]\mathbf{R}(t) = \mathbf{J} \tag{16b}$$

or equivalently,

$$\mathbf{l}_i^{\mathrm{T}}(t)\mathbf{r}_j(t) = \delta_{ij},\ i,j = 1 - 4N, \tag{17a}$$

$$\dot{\mathbf{l}}_i^{\mathrm{T}}(t)\mathbf{r}_j(t) - \mathbf{l}_i^{\mathrm{T}}(t)[\mathbf{A}^{-1}(t)\mathbf{B}(t)]\mathbf{r}_j(t) = \mu_i\delta_{ij}, \tag{17b}$$

where δ_{ij} is the Kronecker delta, the superscript T means the transpose, and, $\mathbf{r}_i(t)$ and $\mathbf{l}_i(t)$ are the ith column vector of $\mathbf{R}(t)$ and $\mathbf{L}(t)$, respectively.

(2) *Structure of the eigenvectors and the adjoint vectors*: Substituting the relation

$$\mathbf{w}(t) = \begin{Bmatrix} \dot{\mathbf{q}}(t) \\ \mathbf{q}(t) \end{Bmatrix} = \mathbf{r}(t)\eta(t)$$

with $\mathbf{q}(t) = \mathbf{u}_{\mathbf{c}}(t)\eta(t)$ and Eq. (12b) into the homogeneous part of Eq. (5), we obtain the relation given by

$$\mathbf{r}(t) = \begin{Bmatrix} \dot{\mathbf{u}}_{\mathbf{c}}(t) + \mu\mathbf{u}_{\mathbf{c}}(t) \\ \mathbf{u}_{\mathbf{c}}(t) \end{Bmatrix}. \tag{18}$$

Likewise, substituting the relation $\mathbf{z}(t) = \bar{\mathbf{l}}(t)\zeta(t)$ with $\bar{\mathbf{l}}(t) = \bar{\mathbf{A}}^{\mathrm{T}}(t)\bar{\boldsymbol{l}}(t)$ and Eq. (14c) into the adjoint Eq. (13), we obtain the relation given by

$$\boldsymbol{l}(t) = \begin{Bmatrix} -\dot{\bar{\mathbf{v}}}_{\mathbf{c}}(t) + \{\mu - [\dot{\mathbf{M}}(t)\mathbf{M}^{-1}(t)]^{\mathrm{T}}\}\bar{\mathbf{v}}_{\mathbf{c}}(t) \\ \bar{\mathbf{v}}_{\mathbf{c}}(t) \end{Bmatrix}. \tag{19}$$

Here, the modal and the adjoint vectors are composed, respectively, of

$$\mathbf{u}_{\mathbf{c}}(t) = \begin{Bmatrix} \mathbf{u}(t) \\ \hat{\mathbf{u}}(t) \end{Bmatrix},\ \mathbf{v}_{\mathbf{c}}(t) = \begin{Bmatrix} \mathbf{v}(t) \\ \hat{\mathbf{v}}(t) \end{Bmatrix}, \tag{20}$$

and for the complex equation of motion as in Eq. (1), it holds, in general,

$$\hat{\mathbf{u}}(t) \neq \bar{\mathbf{u}}(t),\ \hat{\mathbf{v}}(t) \neq \bar{\mathbf{v}}(t). \tag{21}$$

Note that the eigenvector $\mathbf{r}(t)$ (and thus $\mathbf{u}_{\mathbf{c}}(t)$) and the adjoint vector $\boldsymbol{l}(t)$ (and thus $\bar{\mathbf{v}}_{\mathbf{c}}(t)$) are periodically time-varying vectors with the period $T = \pi/\Omega$.

For the time-invariant system, i.e. $\mathbf{r}(t) = \mathbf{r}$, $\bar{\mathbf{l}}(t) = \bar{\mathbf{A}}^{\mathrm{T}}\bar{\boldsymbol{l}}$, $\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{B}(t) = \mathbf{B}$, Eqs. (12a) and (14a) (or equivalently, Eqs. (12b) and (14c)), and, Eqs. (18) and (19) reduce to the form of

$$\mu\mathbf{A}\mathbf{r} = \mathbf{B}\mathbf{r}, \quad \bar{\mu}\bar{\mathbf{A}}^{\mathrm{T}}\bar{\boldsymbol{l}} = \bar{\mathbf{B}}^{\mathrm{T}}\bar{\boldsymbol{l}} \tag{22}$$

and

$$\mathbf{r} = \begin{Bmatrix} \mu\mathbf{u}_{\mathbf{c}} \\ \mathbf{u}_{\mathbf{c}} \end{Bmatrix},\ \boldsymbol{l} = \begin{Bmatrix} \mu\bar{\mathbf{v}}_{\mathbf{c}} \\ \bar{\mathbf{v}}_{\mathbf{c}} \end{Bmatrix},\ \mathbf{u}_{\mathbf{c}} = \begin{Bmatrix} \mathbf{u} \\ \hat{\mathbf{u}} \end{Bmatrix},\ \mathbf{v}_{\mathbf{c}} = \begin{Bmatrix} \mathbf{v} \\ \hat{\mathbf{v}} \end{Bmatrix}, \tag{23}$$

which are consistent with the previous results in Ref. [1].

(3) *Modal equations and eigensolutions*: The complex state and adjoint vectors, $\mathbf{w}(t)$ and $\mathbf{z}(t)$, can be expanded in terms of the eigenvectors and the adjoint vectors, respectively, for the rotor system (5), as

$$\mathbf{w}(t) = \sum_{r=1}^{4N}\{\mathbf{r}(t)\eta(t)\}_r = \sum_{i=B,F}\sum_{r=-N}^{N}\{\mathbf{r}(t)\eta(t)\}_r^i, \tag{24a}$$

$$\mathbf{z}(t) = \sum_{r=1}^{4N}\{\bar{\mathbf{l}}(t)\zeta(t)\}_r = \sum_{i=B,F}\sum_{r=-N}^{N}\{\bar{\mathbf{l}}(t)\zeta(t)\}_r^i, \tag{24b}$$

where the prime notation in the summation implies exclusion of $r=0$, $\eta(t)$ and $\zeta(t)$ are the principal coordinates of the original and adjoint systems, respectively, and the superscripts B and F refer to the backward and forward modes, respectively, following the well-established convention for mode classification in rotor dynamics [1].

Following the notational convention in Eq. (24), the bi-orthonormality condition (17a) can be rewritten as

$$\mathbf{l}_s^{kT}(t)\mathbf{r}_r^i(t) = \delta_{rs}^{ik}, \quad ; r,s = \pm 1, \pm 2, \ldots, \pm N;\ i,k = B,F \tag{17a$'$}$$

Substituting Eq. (24a) into Eq. (5), using relation (12b), pre-multiplying by $\boldsymbol{l}_s^{k\mathrm{T}} = \mathbf{l}_s^{k\mathrm{T}}\mathbf{A}^{-1}(t)$, and, using the bi-orthonormality condition (17a′) and relation (19), we can obtain the $4N$ sets of complex modal equations of motion as

$$\dot{\eta}_r^i(t) = \mu_r^i\eta_r^i(t) + \bar{\mathbf{v}}_{\mathbf{c}r}^{i\mathrm{T}}(t)\mathbf{f}(t) = \mu_r^i\eta_r^i(t) + \bar{\mathbf{v}}_r^{i\mathrm{T}}(t)\mathbf{g}(t) + \bar{\hat{\mathbf{v}}}_r^{i\mathrm{T}}(t)\bar{\mathbf{g}}(t);\ r = \pm 1, \pm 2, \ldots, \pm N; \quad i = B,F. \tag{25}$$

Recalling the Floquet theory that, from the one periodic solution, the entire time response of the eigensolutions can be expressed periodically with the base of that period, we can expand the eigenvector $\mathbf{u}_\mathbf{c}(t)$ and the adjoint vector $\mathbf{v}_\mathbf{c}(t)$ in Eq. (20) by Fourier series as [8]

$$\mathbf{u}_r^i(t) = \sum_{m=-\infty}^{\infty} \mathbf{u}_{r(m)}^i \mathrm{e}^{\mathrm{j}2m\Omega t}, \quad \hat{\mathbf{u}}_r^i(t) = \sum_{m=-\infty}^{\infty} \hat{\mathbf{u}}_{r(m)}^i \mathrm{e}^{\mathrm{j}2m\Omega t}, \tag{26a}$$

$$\mathbf{v}_r^i(t) = \sum_{m=-\infty}^{\infty} \mathbf{v}_{r(m)}^i \mathrm{e}^{\mathrm{j}2m\Omega t}, \quad \hat{\mathbf{v}}_r^i(t) = \sum_{m=-\infty}^{\infty} \hat{\mathbf{v}}_{r(m)}^i \mathrm{e}^{\mathrm{j}2m\Omega t}, \tag{26b}$$

where $\mathbf{u}_{r(m)}^i$, $\hat{\mathbf{u}}_{r(m)}^i$, $\mathbf{v}_{r(m)}^i$ and $\hat{\mathbf{v}}_{r(m)}^i$ are the complex Fourier coefficient vectors associated with the complex harmonic function of $\mathrm{e}^{\mathrm{j}2m\Omega t}$.

From Eqs. (25) and (26), we can obtain the forced response of the general rotor system (5) as

$$\begin{aligned}
\mathbf{p}(t) &= \sum_{i=B,F}\sum_{r=-N}^{N}{}' \{\mathbf{u}(t)\eta(t)\}_r^i \\
&= \sum_{i=B,F}\sum_{r=-N}^{N}{}' \left[\sum_{m=-\infty}^{\infty} \mathbf{u}_{r(m)}^i \mathrm{e}^{\mathrm{j}2m\Omega t}\int_0^t \mathrm{e}^{\mu_r^i(t-\tau)} \sum_{k=-\infty}^{\infty} (\bar{\mathbf{v}}_{r(k)}^{i\mathrm{T}} \mathrm{e}^{-\mathrm{j}2k\Omega\tau}\mathbf{g}(\tau) + \bar{\hat{\mathbf{v}}}_{r(k)}^{i\mathrm{T}} \mathrm{e}^{-\mathrm{j}2k\Omega\tau}\bar{\mathbf{g}}(\tau))\,\mathrm{d}\tau\right] \\
&= \sum_{i=B,F}\sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left\{\int_0^t \mathrm{e}^{(\mu_r^i+\mathrm{j}2m\Omega)(t-\tau)}[\mathbf{u}_{r(m)}^i\bar{\mathbf{v}}_{r(k)}^{i\mathrm{T}}\mathbf{g}(\tau) + \mathbf{u}_{r(m)}^i\bar{\hat{\mathbf{v}}}_{r(k)}^{i\mathrm{T}}\bar{\mathbf{g}}(\tau)]\mathrm{e}^{\mathrm{j}2(m-k)\Omega\tau}\,\mathrm{d}\tau\right\} \\
&= \sum_{i=B,F}\sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\int_0^t \mathrm{e}^{(\mu_r^i+\mathrm{j}2m\Omega)(t-\tau)}[\mathbf{u}_{r(m)}^i\bar{\mathbf{v}}_{r(m-n)}^{i\mathrm{T}}\mathbf{g}(\tau) + \mathbf{u}_{r(m)}^i\bar{\hat{\mathbf{v}}}_{r(m-n)}^{i\mathrm{T}}\bar{\mathbf{g}}(\tau)]\mathrm{e}^{\mathrm{j}2n\Omega\tau}\,\mathrm{d}\tau\right\} \\
&= \sum_{i=B,F}\sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\int_0^t \mathrm{e}^{(\mu_r^i+j2m\Omega)(t-\tau)}[\mathbf{u}_{r(m)}^i\bar{\mathbf{v}}_{r(m-n)}^{i\mathrm{T}}\mathbf{g}_{;n}(\tau) + \mathbf{u}_{r(m)}^i\bar{\hat{\mathbf{v}}}_{r(m-n)}^{i\mathrm{T}}\bar{\mathbf{g}}_{;-n}(\tau)]\,\mathrm{d}\tau\right\}
\end{aligned} \tag{27a}$$

and its complex pair is

$$\hat{\mathbf{p}}(t) = \sum_{i=B,F}\sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\int_0^t \mathrm{e}^{(\mu_r^i+\mathrm{j}2m\Omega)(t-\tau)}[\hat{\mathbf{u}}_{r(m)}^i\bar{\mathbf{v}}_{r(m-n)}^{i\mathrm{T}}\mathbf{g}_{;n}(\tau) + \hat{\mathbf{u}}_{r(m)}^i\bar{\hat{\mathbf{v}}}_{r(m-n)}^{i\mathrm{T}}\bar{\mathbf{g}}_{;-n}(\tau)]\right\}, \tag{27b}$$

where the modulated excitation vector with the complex harmonic function of frequency $2n\Omega$ is defined as

$$\mathbf{g}_{;n}(t) = \mathbf{g}(t)\mathrm{e}^{\mathrm{j}2n\Omega t}. \tag{27c}$$

(4) Direct numerical solution method: In this direct modal analysis approach for the periodically time-varying parameter system (1), the eigenvalues and the corresponding periodically time-varying eigenvectors can be analytically obtained from Eqs. (8) to (17) at least in theory. However, the closed form solutions are limited only to a few simple cases because of mathematical complexity. For most of practical applications, numerical approach is taken instead, as follows. First, the monodromy matrix, $\Phi(T, 0)$, is obtained by numerical integration of Eq. (8) with respect to time for given $\mathbf{A}(t)$, $\mathbf{B}(t)$ and initial condition $\Phi(0,0) = \mathbf{I}_{4N\times 4N}$. Second, the characteristic multipliers ω_i and the corresponding matrix of eigenvectors $\mathbf{R}(0)$ of $\Phi(T, 0)$ are calculated from Eq. (11). Then, the Jordan normal form of matrix $\mathbf{J}$ is formed with its diagonal entries $\mu_i = (1/T)\log(\omega_i)$ and $\mathbf{R}(t)$ can be solved by numerical integration of Eq. (12) with the initial condition $\mathbf{R}(0)$. The same procedure applies to the adjoint matrix $\mathbf{L}(t)$ using $\mathbf{R}(t)$ and the bi-orthonormality conditions (16). Note that, once the periodically time-varying modal (adjoint) vectors are obtained, calculation of the Fourier coefficient modal (adjoint) vectors, which are constant vectors, in Eq. (26) becomes straightforward.
Although the above procedure looks like a novel, analytical approach, one of its critical drawbacks is the numerical instability, since it suffers from serious accumulated error with extensive numerical integration processes [14]. For example, the complex Fourier coefficient modal (adjoint) vectors are very vulnerable to the numerical errors with $\mathbf{R}(t)$ and $\mathbf{L}(t)$. An alternative way of improving numerical accuracy is to develop a direct calculation method of the complex Fourier coefficient modal (adjoint) vectors by constructing the Hill's infinite order matrix as described in Appendix A. The Hill's matrix essentially takes the identical form to Eq. (35a), which will be treated more in details in the following sections.

2.3. Complex modal solution by coordinate transformation

(1) *Equation of motion in the modulated coordinates*: Eq. (1) can be easily transformed to an infinite order matrix equation given by

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{M}}_\mathbf{f} & \bar{\mathbf{M}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{M}_\mathbf{b} & \mathbf{M}_\mathbf{f} & \mathbf{M}_\mathbf{r} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{M}}_\mathbf{r} & \bar{\mathbf{M}}_\mathbf{f} & \bar{\mathbf{M}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{M}_\mathbf{b} & \mathbf{M}_\mathbf{f} & \mathbf{M}_\mathbf{r} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{M}}_\mathbf{r} & \bar{\mathbf{M}}_\mathbf{f} & \bar{\mathbf{M}}_\mathbf{b} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_\mathbf{b} & \mathbf{M}_\mathbf{f} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \ddot{\bar{\mathbf{p}}}\mathrm{e}^{-\mathrm{j}2\Omega t} \\ \ddot{\mathbf{p}}\mathrm{e}^{-\mathrm{j}2\Omega t} \\ \ddot{\bar{\mathbf{p}}} \\ \ddot{\mathbf{p}} \\ \ddot{\bar{\mathbf{p}}}\mathrm{e}^{\mathrm{j}2\Omega t} \\ \ddot{\mathbf{p}}\mathrm{e}^{\mathrm{j}2\Omega t} \\ \vdots \end{Bmatrix}$$

$$+ \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{C}}_\mathbf{f} & \bar{\mathbf{C}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{C}_\mathbf{b} & \mathbf{C}_\mathbf{f} & \mathbf{C}_\mathbf{r} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{C}}_\mathbf{r} & \bar{\mathbf{C}}_\mathbf{f} & \bar{\mathbf{C}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_\mathbf{b} & \mathbf{C}_\mathbf{f} & \mathbf{C}_\mathbf{r} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{C}}_\mathbf{r} & \bar{\mathbf{C}}_\mathbf{f} & \bar{\mathbf{C}}_\mathbf{b} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_\mathbf{b} & \mathbf{C}_\mathbf{f} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \dot{\bar{\mathbf{p}}}\mathrm{e}^{-\mathrm{j}2\Omega t} \\ \dot{\mathbf{p}}\mathrm{e}^{-\mathrm{j}2\Omega t} \\ \dot{\bar{\mathbf{p}}} \\ \dot{\mathbf{p}} \\ \dot{\bar{\mathbf{p}}}\mathrm{e}^{\mathrm{j}2\Omega t} \\ \dot{\mathbf{p}}\mathrm{e}^{\mathrm{j}2\Omega t} \\ \vdots \end{Bmatrix}$$

$$
+\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \bar{\mathbf{K}}_\mathbf{f} & \bar{\mathbf{K}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{K}_\mathbf{b} & \mathbf{K}_\mathbf{f} & \mathbf{K}_\mathbf{r} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \bar{\mathbf{K}}_\mathbf{r} & \bar{\mathbf{K}}_\mathbf{f} & \bar{\mathbf{K}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{K}_\mathbf{b} & \mathbf{K}_\mathbf{f} & \mathbf{K}_\mathbf{r} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{K}}_\mathbf{r} & \bar{\mathbf{K}}_\mathbf{f} & \bar{\mathbf{K}}_\mathbf{b} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_\mathbf{b} & \mathbf{K}_\mathbf{f} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{Bmatrix} \vdots \\ \bar{\mathbf{p}}e^{-j2\Omega t} \\ \mathbf{p}e^{-j2\Omega t} \\ \bar{\mathbf{p}} \\ \mathbf{p} \\ \bar{\mathbf{p}}e^{j2\Omega t} \\ \mathbf{p}e^{j2\Omega t} \\ \vdots \end{Bmatrix}
=
\begin{Bmatrix} \vdots \\ \bar{\mathbf{g}}e^{-j2\Omega t} \\ \mathbf{g}e^{-j2\Omega t} \\ \bar{\mathbf{g}} \\ \mathbf{g} \\ \bar{\mathbf{g}}e^{j2\Omega t} \\ \mathbf{g}e^{j2\Omega t} \\ \vdots \end{Bmatrix}
\tag{28}
$$

Introducing a set of modulated complex coordinate and force vectors, $\mathbf{p}_{;n}$ and $\mathbf{g}_{;n}$, where the modulation index, n, is an arbitrary integer, defined as

$$
\mathbf{p}_{;n}(t) \equiv \mathbf{p}(t)e^{j2n\Omega t},\ \mathbf{g}_{;n}(t) \equiv \mathbf{g}(t)e^{j2n\Omega t},\ \bar{\mathbf{p}}_{;n}(t) \equiv \bar{\mathbf{p}}(t)e^{-j2n\Omega t},\ \bar{\mathbf{g}}_{;n}(t) \equiv \bar{\mathbf{g}}(t)e^{-j2n\Omega t}, \tag{29}
$$

we can rewrite Eq. (28) as

$$
\underset{\sim}{\mathbf{M}}\,\ddot{\underset{\sim}{\mathbf{p}}}(t) + \underset{\sim}{\mathbf{C}}\,\dot{\underset{\sim}{\mathbf{p}}}(t) + \underset{\sim}{\mathbf{K}}\,\underset{\sim}{\mathbf{p}}(t) = \underset{\sim}{\mathbf{g}}(t), \tag{30}
$$

where

$$
\underset{\sim}{\mathbf{M}} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \bar{\mathbf{M}}_\mathbf{f} & \bar{\mathbf{M}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{M}_\mathbf{b} & \mathbf{M}_\mathbf{f} & \mathbf{M}_\mathbf{r} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \bar{\mathbf{M}}_\mathbf{r} & \bar{\mathbf{M}}_\mathbf{f} & \bar{\mathbf{M}}_\mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{M}_\mathbf{b} & \mathbf{M}_\mathbf{f} & \mathbf{M}_\mathbf{r} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{M}}_\mathbf{r} & \bar{\mathbf{M}}_\mathbf{f} & \bar{\mathbf{M}}_\mathbf{b} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_\mathbf{b} & \mathbf{M}_\mathbf{f} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},\quad
\underset{\sim}{\mathbf{C}} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \bar{\mathbf{C}}_{\mathbf{f};1} & \bar{\mathbf{C}}_{\mathbf{b};1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{C}_{\mathbf{b};-1} & \mathbf{C}_{\mathbf{f};-1} & \mathbf{C}_{\mathbf{r};0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \bar{\mathbf{C}}_{\mathbf{r};1} & \bar{\mathbf{C}}_{\mathbf{f};0} & \bar{\mathbf{C}}_{\mathbf{b};0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{\mathbf{b};0} & \mathbf{C}_{\mathbf{f};0} & \mathbf{C}_{\mathbf{r};1} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{C}}_{\mathbf{r};0} & \bar{\mathbf{C}}_{\mathbf{f};-1} & \bar{\mathbf{C}}_{\mathbf{b};-1} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_{\mathbf{b};1} & \mathbf{C}_{\mathbf{f};1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
$$

$$
\underset{\sim}{\mathbf{K}} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \bar{\mathbf{K}}_{\mathbf{f};1} & \bar{\mathbf{K}}_{\mathbf{b};1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{K}_{\mathbf{b};-1} & \mathbf{K}_{\mathbf{f};-1} & \mathbf{K}_{\mathbf{r};0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \bar{\mathbf{K}}_{\mathbf{r};1} & \bar{\mathbf{K}}_{\mathbf{f};0} & \bar{\mathbf{K}}_{\mathbf{b};0} & \mathbf{0} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\mathbf{b};0} & \mathbf{K}_{\mathbf{f};0} & \mathbf{K}_{\mathbf{r};1} & \mathbf{0} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{K}}_{\mathbf{r};0} & \bar{\mathbf{K}}_{\mathbf{f};-1} & \bar{\mathbf{K}}_{\mathbf{b};-1} & \cdots \\
\cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\mathbf{b};1} & \mathbf{K}_{\mathbf{f};1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},\quad
\underset{\sim}{\mathbf{p}}(t) = \begin{Bmatrix} \vdots \\ \bar{\mathbf{p}}_{;1}(t) \\ \mathbf{p}_{;-1}(t) \\ \bar{\mathbf{p}}_{;0}(t) \\ \mathbf{p}_{;0}(t) \\ \bar{\mathbf{p}}_{;-1}(t) \\ \mathbf{p}_{;1}(t) \\ \vdots \end{Bmatrix},\quad
\underset{\sim}{\mathbf{g}}(t) = \begin{Bmatrix} \vdots \\ \bar{\mathbf{g}}_{;1}(t) \\ \mathbf{g}_{;-1}(t) \\ \bar{\mathbf{g}}_{;0}(t) \\ \mathbf{g}_{;0}(t) \\ \bar{\mathbf{g}}_{;-1}(t) \\ \mathbf{g}_{;1}(t) \\ \vdots \end{Bmatrix},
$$

and

$$
\begin{gathered}
\mathbf{C}_{i;n} = \mathbf{C}_i - j4n\Omega\mathbf{M}_i, \\
\mathbf{K}_{i;n} = \mathbf{K}_i - j2n\Omega\mathbf{C}_i - 4n^2\Omega^2\mathbf{M}_i, \\
i = \mathbf{r}, \mathbf{b}, \mathbf{f},\ n = 0, \pm 1, \pm 2, \ldots .
\end{gathered}
$$

Note that the differential equation (1) with periodically time-varying parameters is transformed into Eq. (30) with time-invariant parameters, at the expense of introducing the coordinate vector of infinite dimension.

(2) *Modal analysis*: The equation of motion (30) can be rewritten, in the state space, as

$$\underset{\sim}{\mathbf{A}}\,\dot{\underset{\sim}{\mathbf{w}}}(t) = \underset{\sim}{\mathbf{B}}\,\underset{\sim}{\mathbf{w}}(t) + \underset{\sim}{\mathbf{F}}(t), \tag{31}$$

where

$$\underset{\sim}{\mathbf{A}} = \begin{bmatrix} \underset{\sim}{\mathbf{0}} & \underset{\sim}{\mathbf{M}} \\ \underset{\sim}{\mathbf{M}} & \underset{\sim}{\mathbf{C}} \end{bmatrix}, \quad \underset{\sim}{\mathbf{B}} = \begin{bmatrix} \underset{\sim}{\mathbf{M}} & \underset{\sim}{\mathbf{0}} \\ \underset{\sim}{\mathbf{0}} & -\underset{\sim}{\mathbf{K}} \end{bmatrix}, \quad \underset{\sim}{\mathbf{w}}(t) = \begin{Bmatrix} \dot{\underset{\sim}{\mathbf{p}}}(t) \\ \underset{\sim}{\mathbf{p}}(t) \end{Bmatrix}, \quad \underset{\sim}{\mathbf{F}}(t) = \begin{Bmatrix} \mathbf{0} \\ \underset{\sim}{\mathbf{g}}(t) \end{Bmatrix}.$$

Here, $\mathbf{0}$ represents the zero vector and matrix of infinite dimension. Assuming the solution form of $\underset{\sim}{\mathbf{w}} = \underset{\sim}{\mathbf{r}}\, e^{\lambda t}$, we can obtain the eigenvalue and its adjoint problems associated with Eq. (31) as [1]

$$\lambda^i_{r(m)}\, \underset{\sim}{\mathbf{A}}\, \underset{\sim r(m)}{\mathbf{r}^i} = \underset{\sim}{\mathbf{B}}\, \underset{\sim r(m)}{\mathbf{r}^i} \quad \text{and} \quad \bar{\lambda}^i_{r(m)} \underset{\sim}{\bar{\mathbf{A}}}^{\mathrm{T}}\, \underset{\sim r(m)}{\bar{\boldsymbol{l}}^i} = \underset{\sim}{\bar{\mathbf{B}}}^{\mathrm{T}}\, \underset{\sim r(m)}{\bar{\boldsymbol{l}}^i} \tag{32a}$$

or the equivalent latent value problem as [1]

$$\begin{gathered} \underset{\sim}{\mathbf{D}}(\lambda^i_{r(m)})\underset{\sim cr(m)}{\mathbf{u}^i} = \underset{\sim}{\mathbf{0}} \quad \text{and} \quad \underset{\sim cr(m)}{\bar{\mathbf{v}}^{i\mathrm{T}}}\, \underset{\sim}{\mathbf{D}}(\lambda^i_{r(m)}) = \underset{\sim}{\mathbf{0}}^{\mathrm{T}}, \\ r = \pm 1, \pm 2, \ldots, N, \quad m = 0, \pm 1, \pm 2, \ldots, i = B, F, \end{gathered} \tag{32b}$$

where the lambda matrix of degree two is given by

$$\underset{\sim}{\mathbf{D}}(\lambda) = \lambda^2\, \underset{\sim}{\mathbf{M}} + \lambda\, \underset{\sim}{\mathbf{C}} + \underset{\sim}{\mathbf{K}}$$

and the right and left eigenvectors, and the latent vectors take the form of

$$\underset{\sim}{\mathbf{r}} = \begin{Bmatrix} \lambda \underset{\sim c}{\mathbf{u}} \\ \underset{\sim c}{\mathbf{u}} \end{Bmatrix}, \quad \underset{\sim}{\boldsymbol{l}} = \begin{Bmatrix} \lambda \underset{\sim c}{\bar{\mathbf{v}}} \\ \underset{\sim c}{\bar{\mathbf{v}}} \end{Bmatrix},$$

$$\underset{\sim c}{\mathbf{u}} = \begin{Bmatrix} \cdots & \hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1} & \cdots \end{Bmatrix}^{\mathrm{T}},$$

$$\underset{\sim c}{\mathbf{v}} = \begin{Bmatrix} \cdots & \hat{\mathbf{v}}^{\mathrm{T}}_{;1} & \mathbf{v}^{\mathrm{T}}_{;-1} & \hat{\mathbf{v}}^{\mathrm{T}}_{;0} & \mathbf{v}^{\mathrm{T}}_{;0} & \hat{\mathbf{v}}^{\mathrm{T}}_{;-1} & \mathbf{v}^{\mathrm{T}}_{;1} & \cdots \end{Bmatrix}^{\mathrm{T}}.$$

Here, each pair of eigenvalues, equal in value but different in sign of subscript form a complex conjugate pair. The subscript $r(m)$ refers to the rth eigen (latent) solution in cluster m, as will be shown later (refer to Eq. (40)). Cluster m consists of only the set of eigensolutions associated with the modulation index m, or equivalently, with the shifted eigenvalues by $j2m\Omega$.

The eigenvalues and eigenvectors, obtained from Eq. (32), are normalized so as to satisfy the bi-orthonormality conditions given by

$$\begin{gathered} \underset{\sim r(m)}{\bar{\boldsymbol{l}}^{iT}}\, \underset{\sim}{\mathbf{A}}\, \underset{\sim s(\ell)}{\mathbf{r}^k} = \delta^{ki}_{s(\ell)r(m)} \quad \text{and} \quad \underset{\sim r(m)}{\bar{\boldsymbol{l}}^{iT}}\, \underset{\sim}{\mathbf{B}}\, \underset{\sim s(\ell)}{\mathbf{r}^k} = \lambda^i_{r(m)} \delta^{ki}_{s(\ell)r(m)}, c \\ r, s = \pm 1, \pm 2, \ldots, N, \quad m, \ell = 0, \pm 1, \pm 2, \ldots \quad i,\ k = B, F. \end{gathered} \tag{33}$$

Since the eigensolution takes the form of

$$\underset{\sim}{\mathbf{p}}(t) = \underset{\sim c}{\mathbf{u}}\, e^{\lambda t} = \begin{Bmatrix} \cdots & \hat{\mathbf{u}}^{\mathrm{T}}_{;1} & \mathbf{u}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}^{\mathrm{T}}_{;0} & \mathbf{u}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}^{\mathrm{T}}_{;-1} & \mathbf{u}^{\mathrm{T}}_{;1} & \cdots \end{Bmatrix}^{\mathrm{T}} e^{\lambda t}. \tag{34}$$

Eq. (32b) can be rewritten as

$$\underset{\sim}{\mathbf{D}}(\lambda)\underset{\sim c}{\mathbf{u}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{D}}_{\mathbf{f};1} & \bar{\mathbf{D}}_{\mathbf{b};1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{D}_{\mathbf{b};-1} & \mathbf{D}_{\mathbf{f};-1} & \mathbf{D}_{\mathbf{r};0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{D}}_{\mathbf{r};1} & \bar{\mathbf{D}}_{\mathbf{f};0} & \bar{\mathbf{D}}_{\mathbf{b};0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathbf{b};0} & \mathbf{D}_{\mathbf{f};0} & \mathbf{D}_{\mathbf{r};1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{D}}_{\mathbf{r};0} & \bar{\mathbf{D}}_{\mathbf{f};-1} & \bar{\mathbf{D}}_{\mathbf{b};-1} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathbf{b};1} & \mathbf{D}_{\mathbf{f};1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \hat{\mathbf{u}}_{;1} \\ \mathbf{u}_{;-1} \\ \hat{\mathbf{u}}_{;0} \\ \mathbf{u}_{;0} \\ \hat{\mathbf{u}}_{;-1} \\ \mathbf{u}_{;1} \\ \vdots \end{Bmatrix} = \underset{\sim}{\mathbf{0}}, \tag{35a}$$

$$\underset{\sim c}{\bar{\mathbf{v}}}^{\mathrm{T}}\,\mathbf{D}(\lambda) = \left\{ \cdots \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;1} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;0} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;0} \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;-1} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;1} \quad \cdots \right\}$$
$$\times \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \bar{\mathbf{D}}_{\mathbf{f};1} & \bar{\mathbf{D}}_{\mathbf{b};1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{D}_{\mathbf{b};-1} & \mathbf{D}_{\mathbf{f};-1} & \mathbf{D}_{\mathbf{r};0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \bar{\mathbf{D}}_{\mathbf{r};1} & \bar{\mathbf{D}}_{\mathbf{f};0} & \bar{\mathbf{D}}_{\mathbf{b};0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathbf{b};0} & \mathbf{D}_{\mathbf{f};0} & \mathbf{D}_{\mathbf{r};1} & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{D}}_{\mathbf{r};0} & \bar{\mathbf{D}}_{\mathbf{f};-1} & \bar{\mathbf{D}}_{\mathbf{b};-1} & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{\mathbf{b};1} & \mathbf{D}_{\mathbf{f};1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathbf{0}^{\mathrm{T}}, \tag{35b}$$

where the $N \times N$ block matrices of the Hill's infinite order matrix $\underset{\sim}{\mathbf{D}}(\lambda)$ with $3N$ bandwidth are given as

$$\mathbf{D}_{i;n}(\lambda) = \lambda^2\mathbf{M}_i + \lambda\mathbf{C}_{i;n} + \mathbf{K}_{i;n}; \quad i = \mathbf{r}, \mathbf{b}, \mathbf{f}; \quad n = 0, \pm 1, \pm 2, \ldots . \tag{35c}$$

And the bi-orthonormality conditions (33) reduces, for $i = k$, $r(m) = s(\ell)$,[1] to [1]

$$\underset{\sim c}{\bar{\mathbf{v}}}^{\mathrm{T}}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\underset{\sim}{\mathbf{D}}(\lambda)\right]\underset{\sim c}{\mathbf{u}} = \{ \cdots \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;1} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;-1} \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;0} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;0} \quad \hat{\bar{\mathbf{v}}}^{\mathrm{T}}_{;-1} \quad \bar{\mathbf{v}}^{\mathrm{T}}_{;1} \quad \cdots \}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\underset{\sim}{\mathbf{D}}(\lambda)\right]\begin{Bmatrix} \vdots \\ \hat{\mathbf{u}}_{;1} \\ \mathbf{u}_{;-1} \\ \hat{\mathbf{u}}_{;0} \\ \mathbf{u}_{;0} \\ \hat{\mathbf{u}}_{;-1} \\ \mathbf{u}_{;1} \\ \vdots \end{Bmatrix} = 1. \tag{36}$$

[1]The relation $r(m) = s(\ell)$ means that $r = s$ and $m = \ell$.

In the above expressions (34), (35) and (36), the sub- and superscripts are omitted for notational simplicity. From Eq. (34), we obtain the relations

$$\mathbf{p}_{;n}(t) = \mathbf{u}^i_{r(m);n} e^{\lambda^i_{r(m)} t}, \quad \bar{\mathbf{p}}_{;n}(t) = \bar{\mathbf{u}}^k_{s(\ell);n} e^{\bar{\lambda}^k_{s(\ell)} t} = \hat{\mathbf{u}}^i_{r(m);n} e^{\lambda^i_{r(m)} t} \tag{37a}$$

implying that

$$\bar{\mathbf{u}}^k_{s(\ell);n} = \hat{\mathbf{u}}^i_{r(m);n}, \ \lambda^i_{r(m)} = \bar{\lambda}^k_{s(\ell)}. \tag{37b}$$

Recall that the parenthesized subscripts denote the modulation indices. Since it holds $i = k$, $s = -r$, $\ell = -m$ from the above relations, we obtain, for every pair of latent vectors $(\hat{\mathbf{u}}^{\mathrm{T}}_{;n}, \mathbf{u}^{\mathrm{T}}_{;n})^{\mathrm{T}}$ associated with the eigenvalue $\lambda^i_{r(m)}$,

$$\bar{\mathbf{u}}^i_{-r(-m);n} = \hat{\mathbf{u}}^i_{r(m);n}, \ \lambda^i_{r(m)} = \bar{\lambda}^i_{-r(-m)}. \tag{38}$$

On the other hand, since it also holds

$$\mathbf{p}_{;n}(t) = \mathbf{p}_{;0}(t) e^{j2n\Omega t} = \mathbf{u}^i_{s(\ell);0} e^{\lambda^i_{s(\ell)} t} e^{j2n\Omega t} = \mathbf{u}^i_{r(m);n} e^{\lambda^i_{r(m)} t} \tag{39a}$$

we obtain the circulation formula between the eigenvalues associated with the latent vectors $\mathbf{u}_{;n}$ and $\mathbf{u}_{;0}$ as

$$\lambda^i_{r(m)} = \lambda^i_{s(\ell)} + j2n\Omega, \quad \lambda^i_{-r(-m)} = \bar{\lambda}^i_{r(m)} = \lambda^i_{-s(-\ell)} - j2n\Omega = \overline{\lambda^i_{s(\ell)} + j2n\Omega}, \tag{39b}$$

which means that, unless $\tilde{\mathbf{A}}$ is a singular matrix ($\mathbf{M_f}$ and thus $\tilde{\mathbf{A}}$ seldom become singular), the shifted eigenvalue by $j2n\Omega$ and its complex conjugate also become eigenvalues for any integer value of n. Note that the subscripts r, s, m, ℓ can be arbitrarily assigned. However, it will prove very convenient to assign the subscripts such that $m = n$, $r = s$, and $\ell = 0$, i.e.

$$\lambda^i_{r(n)} = \lambda^i_{r(0)} + j2n\Omega, \quad \lambda^i_{-r(-n)} = \bar{\lambda}^i_{r(n)} = \lambda^i_{-r(0)} - j2n\Omega = \overline{\lambda^i_{r(0)} + j2n\Omega}. \tag{40}$$

The above relations hold only between the same rotational modes, forward or backward. For example, if λ^F_r is an eigenvalue in cluster 0, $\bar{\lambda}^F_r$, $\lambda^F_r + j2n\Omega$, and $\overline{\lambda^F_r + j2n\Omega} = \bar{\lambda}^F_r - j2n\Omega$ also become eigenvalues belonging to cluster 0, n, and $-n$, respectively. In this case, since $\lambda'^F_r = \overline{\lambda^F_r - j2n\Omega}$ is an eigenvalue in cluster n, we can derive the relations such as $\bar{\lambda}'^F_r = \lambda^F_r - j2n\Omega$ (cluster $-n$), $\lambda'^F_r - j2n\Omega = \overline{\lambda^F_r - j2n\Omega} - j2n\Omega = \bar{\lambda}^F_r$ and $\lambda'^F_r = \overline{\overline{\lambda^F_r - j2n\Omega}} + j2n\Omega = \lambda^F_r$. In other words, there exists a circulation relation between the eigenvalues with the shift of $\pm j2n\Omega$. The structure of the eigenvalues and latent vectors can be summarized as follows.

Eigenvalues:

$$\begin{aligned}
\{\lambda^i_{r(m)}\} &= \{\ldots, (\text{cluster} - n), \ldots, (\text{cluster } 0), \ldots, (\text{cluster } n), \ldots\} \\
&= \{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&\quad (\ldots, \bar{\lambda}^F_{N(0)} - j2n\Omega, \bar{\lambda}^B_{N(0)} - j2n\Omega, \ldots, \bar{\lambda}^F_{1(0)} - j2n\Omega, \bar{\lambda}^B_{1(0)} - j2n\Omega, \\
&\quad \lambda^F_{1(0)} - j2n\Omega, \lambda^B_{1(0)} - j2n\Omega, \ldots, \lambda^F_{N(0)} - j2n\Omega, \lambda^B_{N(0)} - j2n\Omega, \ldots), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&\quad (\ldots, \bar{\lambda}^F_{N(0)}, \bar{\lambda}^B_{N(0)}, \ldots, \bar{\lambda}^F_{1(0)}, \bar{\lambda}^B_{1(0)}, \lambda^F_{1(0)}, \lambda^B_{1(0)}, \ldots \lambda^F_{N(0)}, \lambda^B_{N(0)}, \ldots), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

$$(\ldots, \bar{\lambda}^{F}_{N(0)} + j2n\Omega, \bar{\lambda}^{B}_{N(0)} + j2n\Omega, \ldots, \bar{\lambda}^{F}_{1(0)} + j2n\Omega, \bar{\lambda}^{B}_{1(0)} + j2n\Omega,$$
$$\lambda^{F}_{1(0)} + j2n\Omega, \lambda^{B}_{1(0)} + j2n\Omega, \ldots, \lambda^{F}_{N(0)} + j2n\Omega, \lambda^{B}_{N(0)} + j2n\Omega, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\} \tag{41a}$$

Latent vectors:

$$\underset{\sim c}{\mathbf{u}} = \left\{ \cdots \quad \bar{\mathbf{u}}^{iT}_{-r(-m);1} \quad \mathbf{u}^{iT}_{r(m);-1} \quad \bar{\mathbf{u}}^{iT}_{-r(-m);0} \quad \mathbf{u}^{iT}_{r(m);0} \quad \bar{\mathbf{u}}^{iT}_{-r(-m);-1} \quad \mathbf{u}^{iT}_{r(m);1} \quad \cdots \right\}^{T},$$
$$\underset{\sim c}{\mathbf{v}} = \left\{ \cdots \quad \bar{\mathbf{v}}^{iT}_{-r(-m);1} \quad \mathbf{v}^{iT}_{r(m);-1} \quad \bar{\mathbf{v}}^{iT}_{-r(-m);0} \quad \mathbf{v}^{iT}_{r(m);0} \quad \bar{\mathbf{v}}^{iT}_{-r(-m);-1} \quad \mathbf{v}^{iT}_{r(m);1} \quad \cdots \right\}^{T}, \tag{41b}$$

where we use the relation for the left eigenvectors derived similarly to the right eigenvectors, i.e.

$$\bar{\mathbf{v}}^{i}_{-r(-m);n} = \hat{\mathbf{v}}^{i}_{r(m);n}. \tag{41c}$$

(3) *Modal equations and eigensolutions*: The complex state vector, $\underset{\sim}{\mathbf{w}}(t)$, can be expanded as

$$\underset{\sim}{\mathbf{w}}(t) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' [\underset{\sim}{\mathbf{r}}\, \eta(t)]^{i}_{r(m)} \tag{42a}$$

or equivalently,

$$\underset{\sim}{\mathbf{p}}(t) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' [\underset{\sim}{\mathbf{u}}\, \eta(t)]^{i}_{r(m)} \tag{42b}$$

and

$$\mathbf{p}(t) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' [\mathbf{u}_{;0}\eta(t)]^{i}_{r(m)}, \tag{42c}$$

where $\eta(t)$ is the principal coordinate.

Substituting Eq. (42c) into Eq. (31) and using the bi-orthonormality conditions (33), we can obtain the infinite set of complex modal equations of motion as

$$\dot{\eta}^{i}_{r(m)} = \lambda^{i}_{r(m)}\eta^{i}_{r(m)} + \underset{\sim cr(m)}{\bar{\mathbf{v}}^{iT}}\, \underset{\sim}{\mathbf{g}}(t),\ r = \pm 1, \pm 2, \ldots, \pm N; \quad i = B, F; \quad m = 0, \pm 1, \pm 2, \ldots. \tag{43}$$

From Eqs. (42) and (43), we can then obtain the forced response of the general rotor system (1) as

$$\begin{aligned}\mathbf{p}(t) &= \sum_{i=B,F} \sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty} \int_0^t e^{\lambda^{i}_{r(m)}(t-\tau)} \mathbf{u}^{i}_{r(m);0} \underset{\sim cr(m)}{\bar{\mathbf{v}}^{iT}}\, \underset{\sim}{\mathbf{g}}(\tau)\, d\tau \\ &= \sum_{i=B,F} \sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ \int_0^t e^{\lambda^{i}_{r(m)}(t-\tau)} [\mathbf{u}^{i}_{r(m);0} \bar{\mathbf{v}}^{iT}_{r(m);n} \mathbf{g}(\tau) + \mathbf{u}^{i}_{r(m);0} \bar{\hat{\mathbf{v}}}^{iT}_{r(m);n} \bar{\mathbf{g}}(\tau)] e^{j2n\Omega\tau}\, d\tau \right\} \\ &= \sum_{i=B,F} \sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ \int_0^t e^{\lambda^{i}_{r(m)}(t-\tau)} \left[\mathbf{u}^{i}_{r(m);0} \bar{\mathbf{v}}^{iT}_{r(m);n} \mathbf{g}_{;n}(\tau) + \mathbf{u}^{i}_{r(m);0} \bar{\hat{\mathbf{v}}}^{iT}_{r(m);n} \bar{\mathbf{g}}_{;-n}(\tau) \right] d\tau \right\}. \end{aligned} \tag{44}$$

Note that, for the time-invariant system, the index for cluster, m, is confined to zero only, i.e. $m = 0$, which is consistent with the previous results in Ref. [1].

2.4. Relationship between modal solutions by Floquet and coordinate transform approaches

By comparing Eqs. (27) and (44), we can easily derive the relationship between the eigenvalues obtained from the two different approaches using the Floquet theory and the coordinate transformation as

$$\lambda^i_{r(m)} \equiv \mu^i_r + j2m\Omega \tag{45a}$$

and we find the equivalent relationship between the corresponding eigenvectors as

$$\mathbf{u}^i_{r(m-n)} \leftrightarrow \mathbf{u}^i_{r(m);n}, \tag{45b}$$

$$\mathbf{v}^i_{r(m-n)} \leftrightarrow \mathbf{v}^i_{r(m);n}. \tag{45c}$$

From Eq. (45a), we obtain, for $m = 0$, the relation $\lambda^i_{r(0)} \equiv \mu^i_r$, implying that the eigenvalues belonging to cluster 0 (basic cluster) correspond to those associated with the periodically time-varying eigenvectors obtained using Floquet theory and the eigenvalues belonging to cluster m, $\lambda^i_{r(m)}$, consist of the basic eigenvalues that are shifted by $j2m\Omega$. Eqs. (45b), (45c) with $n = 0$ indicates that the right (left) latent vectors, $\mathbf{u}^i_{r(m);0}(\mathbf{v}^i_{r(m);0})$, associated with the eigenvalue, $\lambda^i_{r(m)}$, and the original excitation force, $\mathbf{g}^{;0}(t) = \mathbf{g}(t)$, is nothing but the complex Fourier coefficient vector, $\mathbf{u}^i_{r(m)}(\mathbf{v}^i_{r(m)})$, associated with the complex harmonic function $e^{j2m\Omega t}$, of the periodically time-varying eigenvector $\mathbf{u}^i_r(t)(\mathbf{v}^i_r(t))$ associated with the eigenvalue $\mu^i_r = \lambda^i_{r(0)}$.

The eigenvalues obtained from Floquet theory coincide with the eigenvalues belonging to cluster 0 obtained from coordinate transformation. The infinite set of Fourier coefficient vectors associated with $\mu^i_r = \lambda^i_{r(0)}$ correspond to the eigenvectors associated with $\lambda^i_{r(m)} = \lambda^i_{r(0)} + j2m\Omega$. Eq. (45b) and (45c) with $n \neq 0$ suggests that the left (right) latent vector, $\mathbf{v}^i_{r(m);n}(\mathbf{u}^i_{r(m);n})$, associated with the eigenvalue, $\lambda^i_{r(m)}$, and the modulated excitation force, $\mathbf{g}_{;n}(t) = \mathbf{g}(t)e^{j2n\Omega t}$, corresponds to the Fourier coefficient vector, $\mathbf{v}^i_{r(m-n)}$. This rather intricate relationships (45b) and (45c) may cause a significant discrepancy between two different approaches in numerical calculation of the eigensolutions, because the way of order reduction with the Hill's matrix may be different, using different sets of Ritz vectors for approximation, except the case with the modulation index $n = 0$ or with use of a sufficiently high-order Hill's matrix. Floquet approach does not account for the modulated excitation forces, unlike the coordinate transform approach. In other words, Floquet approach is strictly based on the periodic time response of the system, whereas the coordinate transformation is essentially based on the frequency response of the system.

3. Infinite order directional frequency response matrix (dFRM)

Fourier transforming Eq. (42b), we obtain

$$\underset{\sim}{\mathbf{P}}(j\omega) = \underset{\sim}{\mathbf{H}}(j\omega)\,\underset{\sim}{\mathbf{G}}(j\omega), \tag{46a}$$

where the directional frequency response matrix (dFRM) of infinite order is given as

$$\underset{\sim}{\mathbf{H}}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\underset{\sim}{\mathbf{u}}\,\underset{\sim}{\bar{\mathbf{v}}}^{\mathbf{T}}}{j\omega - \lambda} \right]^i_{r(m)}$$

$$
= \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \frac{1}{j\omega - \lambda^{i}_{r(m)}} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \hat{\mathbf{u}}_{;1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \hat{\mathbf{u}}_{;1}\bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}_{;1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}_{;1}\bar{\mathbf{v}}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}_{;1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}_{;1}\bar{\mathbf{v}}^{\mathrm{T}}_{;1} & \cdots \\ \cdots & \mathbf{u}_{;-1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \mathbf{u}_{;-1}\bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \mathbf{u}_{;-1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \mathbf{u}_{;-1}\bar{\mathbf{v}}^{\mathrm{T}}_{;0} & \mathbf{u}_{;-1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \mathbf{u}_{;-1}\bar{\mathbf{v}}^{\mathrm{T}}_{;1} & \cdots \\ \cdots & \hat{\mathbf{u}}_{;0}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \hat{\mathbf{u}}_{;0}\bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}_{;0}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}_{;0}\bar{\mathbf{v}}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}_{;0}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}_{;0}\bar{\mathbf{v}}^{\mathrm{T}}_{;1} & \cdots \\ \cdots & \mathbf{u}_{;0}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \mathbf{u}_{;0}\bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \mathbf{u}_{;0}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \mathbf{u}_{;0}\bar{\mathbf{v}}^{\mathrm{T}}_{;0} & \mathbf{u}_{;0}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \mathbf{u}_{;0}\bar{\mathbf{v}}^{\mathrm{T}}_{;1} & \cdots \\ \cdots & \hat{\mathbf{u}}_{;-1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \hat{\mathbf{u}}_{;-1}\bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}_{;-1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}_{;-1}\bar{\mathbf{v}}^{\mathrm{T}}_{;0} & \hat{\mathbf{u}}_{;-1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \hat{\mathbf{u}}_{;-1}\bar{\mathbf{v}}^{\mathrm{T}}_{;1} & \cdots \\ \cdots & \mathbf{u}_{;1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;1} & \mathbf{u}_{;1}\bar{\mathbf{v}}^{\mathrm{T}}_{;-1} & \mathbf{u}_{;1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;0} & \mathbf{u}_{;1}\bar{\mathbf{v}}^{\mathrm{T}}_{;0} & \mathbf{u}_{;1}\bar{\hat{\mathbf{v}}}^{\mathrm{T}}_{;-1} & \mathbf{u}_{;1}\bar{\mathbf{v}}^{\mathrm{T}}_{;1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^{i}_{r(m)} \tag{46b}
$$

Here, $\underset{\sim}{\mathbf{P}}(j\omega)$ and $\underset{\sim}{\mathbf{G}}(j\omega)$ are the Fourier transforms of $\underset{\sim}{\mathbf{P}}(t)$ and $\underset{\sim}{\mathbf{g}}(t)$, respectively. Although an infinite number of block dFRMs in Eq. (46b) exist, not all of them are independently determined. For example, it holds, for arbitrary integers k, ℓ and n,

$$
\begin{aligned}
\mathbf{H}_{\mathbf{g}_{;k}\mathbf{p}_{;\ell}}(j\omega - j2n\Omega) &= \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}_{;\ell}\bar{\mathbf{v}}^{\mathrm{T}}_{;-k}}{j\omega - j2n\Omega - \lambda}\right]^{i}_{r(m)} \\
&= \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^{i}_{r(m-n);(\ell+n)}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(m-n);(-k-n)}}{j\omega - \lambda^{i}_{r(m-n)}}\right] \\
&= \sum_{(m-n)=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}_{;(\ell+n)}\bar{\mathbf{v}}^{\mathrm{T}}_{;(-k-n)}}{j\omega - \lambda}\right]^{i}_{r(m-n)} = \mathbf{H}_{\mathbf{g}_{;(k+n)}\mathbf{p}_{;(\ell+n)}}(j\omega)
\end{aligned} \tag{47}
$$

implying that $\mathbf{H}_{\mathbf{g}_{;(k+n)}\mathbf{p}_{;(\ell+n)}}(j\omega)$ can be obtained by shifting $\mathbf{H}_{\mathbf{g}_{;k}\mathbf{p}_{;\ell}}(j\omega)$ by $2n\Omega$ in the frequency domain. In particular, the diagonal block matrices satisfy the relation $\mathbf{H}_{\mathbf{g}_{;n}\mathbf{p}_{;n}}(j\omega) = \mathbf{H}_{\mathbf{g}_{;0}\mathbf{p}_{;0}}(j\omega - j2n\Omega)$. Similar relations can be derived for $\mathbf{H}_{\bar{\mathbf{g}}_{;k}\mathbf{p}_{;\ell}}(j\omega)$, $\mathbf{H}_{\mathbf{g}_{;k}\bar{\mathbf{p}}_{;\ell}}(j\omega)$ and $\mathbf{H}_{\bar{\mathbf{g}}_{;k}\bar{\mathbf{p}}_{;\ell}}(j\omega)$, which will not be repeated here. However, it is concluded that a single row (or column) vector of the infinite order dFRM forms a set of independent dFRMs, which can be written as

$$
\begin{aligned}
\mathbf{P}(j\omega) &= \sum_{n=-\infty}^{\infty} \left\{ \left[\sum_{i=B,F} \sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty} \frac{\mathbf{u}^{i}_{r(m);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(m);n}}{j\omega - \lambda^{i}_{r(m)}}\right] \mathbf{G}_{;n}(j\omega) + \left[\sum_{i=B,F} \sum_{r=-N}^{N}{}' \sum_{m=-\infty}^{\infty} \frac{\mathbf{u}^{i}_{r(m);0}\bar{\hat{\mathbf{v}}}^{i\mathrm{T}}_{r(m);n}}{j\omega - \lambda^{i}_{r(m)}}\right] \hat{\mathbf{G}}_{;-n}(j\omega) \right\} \\
&= \sum_{n=-\infty}^{\infty} \left\{ \mathbf{H}_{\mathbf{g}_{;-n}\mathbf{p}}(j\omega)\mathbf{G}_{;n}(j\omega) + \mathbf{H}_{\bar{\mathbf{g}}_{;-n}\mathbf{p}}(j\omega)\hat{\mathbf{G}}_{;-n}(j\omega) \right\},
\end{aligned} \tag{48}
$$

where the Fourier transforms of the modulated excitation vectors are given by

$$
\mathbf{G}_{;n}(j\omega) = \mathbf{G}\{j(\omega - 2n\Omega)\}, \quad \hat{\mathbf{G}}_{;n}(j\omega) = \hat{\mathbf{G}}\{j(\omega + 2n\Omega)\}.
$$

Here, $\mathbf{G}(j\omega)$ and $\hat{\mathbf{G}}(j\omega)$ are the Fourier transforms of $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively. Note that Eq. (48) can also be derived by direct Fourier transform of Eq. (44). Although there are still an infinite number of dFRMs in Eq. (48), we introduce four dFRMs, that are important in characterizing the system asymmetry and anisotropy, as

$$
\mathbf{H}_{\mathbf{gp}}(j\omega) = \mathbf{H}_{\mathbf{g}_{;0}\mathbf{p}_{;0}}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^{i}_{r(m);0}\bar{\mathbf{v}}^{i\mathrm{T}}_{r(m);0}}{j\omega - \lambda^{i}_{r(m)}}\right],
$$

$$\mathbf{H}_{\hat{\mathbf{g}}p}(j\omega) = \mathbf{H}_{\bar{\mathbf{g}};0\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m);0} \mathbf{v}^{iT}_{-r(-m);0}}{j\omega - \lambda^i_{r(m)}} \right],$$

$$\mathbf{H}_{\tilde{\mathbf{g}}p}(j\omega) = \mathbf{H}_{\bar{\mathbf{g}};-1\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m);0} \mathbf{v}^{iT}_{-r(-m);-1}}{j\omega - \lambda^i_{r(m)}} \right],$$

$$\mathbf{H}_{\breve{\mathbf{g}}p}(j\omega) = \mathbf{H}_{\mathbf{g};-1\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m);0} \bar{\mathbf{v}}^{iT}_{r(m);1}}{j\omega - \lambda^i_{r(m)}} \right]. \tag{49}$$

Here, $\mathbf{H}_{\mathbf{gp}}(j\omega)$ is referred to as the normal dFRM that represents the system symmetry, $\mathbf{H}_{\hat{\mathbf{g}}p}(j\omega)$ is referred to as the reverse dFRM that represents the effect of system anisotropy, and, $\mathbf{H}_{\tilde{\mathbf{g}}p}(j\omega)$ and $\mathbf{H}_{\breve{\mathbf{g}}p}(j\omega)$ are referred to as the modulated dFRMs that represent the effect of system asymmetry and the coupled effect of system anisotropy and asymmetry, respectively.

Similar relations to Eqs. (48) and (49) can also be derived from Eq. (27) as

$$\begin{aligned} \mathbf{P}(j\omega) &= \sum_{n=-\infty}^{\infty} \left\{ \left[\sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \frac{\mathbf{u}^i_{r(m)} \bar{\mathbf{v}}^{iT}_{r(m-n)}}{j(\omega - 2m\Omega) - \mu^i_r} \right] \mathbf{G}_{;n}(j\omega) + \left[\sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \frac{\mathbf{u}^i_{r(m)} \bar{\hat{\mathbf{v}}}^{iT}_{r(m-n)}}{j(\omega - 2m\Omega) - \mu^i_r} \right] \hat{\mathbf{G}}_{;-n}(j\omega) \right\} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \mathbf{H}_{\mathbf{g};-n\mathbf{p}}(j\omega) \mathbf{G}_{;n}(j\omega) + \mathbf{H}_{\bar{\mathbf{g}};-n\mathbf{p}}(j\omega) \hat{\mathbf{G}}_{;-n}(j\omega) \right\}, \end{aligned} \tag{50}$$

and

$$\mathbf{H}_{\mathbf{gp}}(j\omega) = \mathbf{H}_{\mathbf{g};0\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m)} \bar{\mathbf{v}}^{iT}_{r(m)}}{j(\omega - 2m\Omega) - \mu^i_r} \right],$$

$$\mathbf{H}_{\hat{\mathbf{g}}\,\mathbf{p}}(j\omega) = \mathbf{H}_{\bar{\mathbf{g}};0\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m)} \mathbf{v}^{iT}_{-r(-m)}}{j(\omega - 2m\Omega) - \mu^i_r} \right],$$

$$\mathbf{H}_{\tilde{\mathbf{g}}\,\mathbf{p}}(j\omega) = \mathbf{H}_{\bar{\mathbf{g}};-1\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m)} \mathbf{v}^{iT}_{-r(-m+1)}}{j(\omega - 2m\Omega) - \mu^i_r} \right],$$

$$\mathbf{H}_{\breve{\mathbf{g}}\,\mathbf{p}}(j\omega) = \mathbf{H}_{\mathbf{g};-1\mathbf{p};0}(j\omega) = \sum_{m=-\infty}^{\infty} \sum_{i=B,F} \sum_{r=-N}^{N}{}' \left[\frac{\mathbf{u}^i_{r(m)} \bar{\mathbf{v}}^{iT}_{r(m-1)}}{j(\omega - 2m\Omega) - \mu^i_r} \right]. \tag{51}$$

It can be easily shown, using the relations (45), that the theoretical expressions for the dFRMs, Eqs. (49) and (51), derived from the coordinate transformation and the Floquet theory, respectively, are identical. However, the numerical procedures for the dFRM estimates are different from each other. In particular, the truncation schemes of the infinite series expansion (or equivalently, the infinite summation) with respect to the index m are different. For example, truncation of the infinite summation is done with the complex Fourier-series expansion of the periodically time-varying eigenvectors, Eq. (26), for the Floquet approach, whereas the order reduction is done with the Hill's infinite order matrix, Eq. (35), for the coordinate transform approach.

4. Numerical example

This section demonstrates and compares the two different modal analysis methods developed previously with a simple, yet general rotor system model, which consists of a rigid rotor with asymmetric mass moments of inertia, a mass-less shaft with asymmetric shaft stiffness, and two orthotropic bearings. The detailed

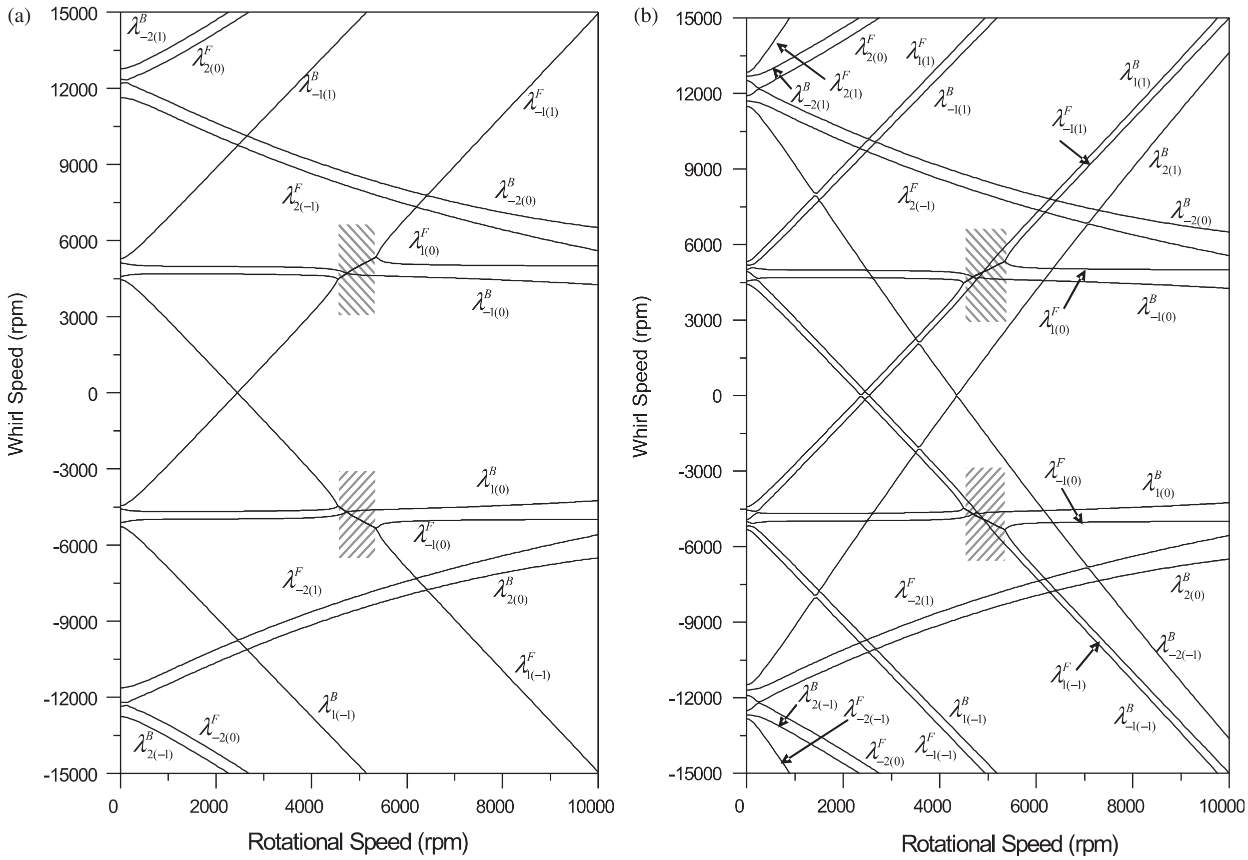

Fig. 2. Whirl speed charts from the reduced Hill's matrix of order: (a) $4N$ and (b) $6N$; unstable region is hatched ($\delta = \Delta = 0.3$).

descriptions of the rotor model are treated in Appendix B. The physical data of the model (refer to Fig. 1) used in the simulations are

$$\text{Disk}: \rho = 7850\,\text{kg/m}^3, \quad D = 400\,\text{mm}, \quad l_D = 30\,\text{mm}, \quad M = \rho\pi l_D D^2/4 = 29.6\,\text{kg},$$
$$J_p = \rho M D^2/8 = 0.5918\,\text{kg}\,\text{m}^2, \quad J = \rho M(3D^2/4 + l_D^2)/12 = 0.298\,\text{kg}\,\text{m}^2.$$

$$\text{Shaft}: E = 2.07 \times 10^{11}\,\text{N/m}^2, \quad L_1 = 0.2\,\text{m}, \quad L_2 = 0.3\,\text{m}, \quad L = 0.5\,\text{m}, \quad d = 40\,\text{mm}, \quad I = \pi d^4/64,$$
$$I_{\xi_1} = I_{\xi_2} = I(1+\delta), \quad I_{\eta_1} = I_{\eta_2} = I(1-\delta), \quad k_r = 1.1 \times 10^7\,\text{N/m}, \quad k_\theta = 5.85 \times 10^5\,\text{N/rad},$$
$$k_{r\theta} = -1.46 \times 10^6\,\text{N/rad}\,\text{m}, \quad \Delta k_{rr} = \delta k_{rr}, \quad \Delta k_\theta = \delta k_\theta, \quad \Delta k_{r\theta} = \delta k_{r\theta},$$
$$c_r = 50\,\text{Ns/m}, \quad c_\theta = 30\,\text{N}\,\text{s/rad}.$$

$$\text{Bearings}: k_{b1} = k_{b2} = k_b = 2 \times 10^7\,\text{N/m}, \quad \Delta k_{b1} = \Delta k_{b2} = \Delta k_b, \quad c_{b1} = c_{b2} = 3{,}000\,\text{Ns/m},$$
$$\Delta c_{b1} = \Delta c_{b2} = 0.$$

Figs. 2(a) and (b) show the whirl speeds calculated from the reduced Hill's matrix of order $4N$ ($N = 4$ for the analysis model) and $6N$, respectively. The unstable region (hatched in the figures), due to the presence of rotating asymmetry, exists between 4500 and 5450 rev/min. Table 1 compares the eigenvalues calculated at the rotational speed of 10,000 rev/min from the reduced Hill's matrix of order $4N$ and $6N$. And it clearly identifies the original modes, the modulated modes, and the complex conjugate modes. Note that, as the order of the reduced Hill's matrix increases, the estimation accuracy of eigenvalues improves, but the reduced Hill's matrix

Table 1
Eigenvalues of the analysis rotor model @10,000 rev/min: $\delta = \Delta = 0.3$

Coordinate Transform Method		Mode Classification	Mode Equivalence by Floquet Method	Remark
Hill's matrix of order $4N^*$	Hill's matrix of order $6N$			
$-70.7+j\,2685.4$	$-69.9+j\,2683.8$	$\lambda^F_{2(0)}$	μ^F_2	cluster 0 (basic modes)
$-70.7-j\,2685.4$	$-69.9-j\,2683.8$	$\lambda^F_{-2(0)}$	$\mu^F_{-2}(\bar{\mu}^F_2)$	
$-49.3+j\,683.8$	$-49.2+j\,682.5$	$\lambda^B_{-2(0)}$	$\mu^B_{-2}(\bar{\mu}^B_2)$	
$-49.3-j\,683.8$	$-49.2-j\,682.5$	$\lambda^B_{2(0)}$	μ^B_2	
$-5.6+j\,523.7$	$-5.6+j\,523.6$	$\lambda^F_{1(0)}$	μ^F_1	
$-5.6-j\,523.7$	$-5.6-j\,523.6$	$\lambda^F_{-1(0)}$	$\mu^F_{-1}(\bar{\mu}^F_1)$	
$-25.7+j\,445.8$	$-25.7+j\,445.9$	$\lambda^B_{-1(0)}$	$\mu^B_{-1}(\bar{\mu}^B_1)$	
$-25.7-j\,445.8$	$-25.7-j\,445.9$	$\lambda^B_{1(0)}$	μ^B_1	
—	$-81.7+j\,4788.5$	$\lambda^F_{2(1)}$	$\mu^F_2+j2\Omega$	cluster 1 (modulated modes)
$-64.8-j\,592.7$	$-69.8-j\,589.3$	$\lambda^F_{-2(1)}$	$\mu^F_{-2}+j2\Omega$	
$-48.2+j\,2784.4$	$-48.9+j\,2777.2$	$\lambda^B_{-2(1)}$	$\mu^B_{-2}+j2\Omega$	
—	$-32.4+j\,1426.5$	$\lambda^B_{2(1)}$	$\mu^B_2+j2\Omega$	
—	$-7.2+j\,2628.7$	$\lambda^F_{1(1)}$	$\mu^F_1+j2\Omega$	
$-2.7+j\,1564.3$	$-5.8+j\,1570.3$	$\lambda^F_{-1(1)}$	$\mu^F_{-1}+j2\Omega$	
$-26.2+j\,2543.8$	$-25.8+j\,2540.3$	$\lambda^B_{-1(1)}$	$\mu^B_{-1}+j2\Omega$	
—	$-18.8+j\,1623.1$	$\lambda^B_{1(1)}$	$\mu^B_1+j2\Omega$	
$-64.8+j\,592.7$	$-69.8+j\,589.3$	$\lambda^F_{2(-1)}$	$\mu^F_2-j2\Omega$	cluster-1 (modulated modes)
—	$-81.7-j\,4788.5$	$\lambda^F_{-2(-1)}$	$\mu^F_{-2}-j2\Omega$	
—	$-32.4-j\,1426.5$	$\lambda^B_{-2(-1)}$	$\mu^B_{-2}-j2\Omega$	
$-48.2-j\,2784.4$	$-48.9-j\,2777.2$	$\lambda^B_{2(-1)}$	$\mu^B_2-j2\Omega$	
$-2.7-j\,1564.3$	$-5.8-j\,1570.3$	$\lambda^F_{1(-1)}$	$\mu^F_1-j2\Omega$	
—	$-7.2-j\,2628.7$	$\lambda^F_{-1(-1)}$	$\mu^F_{-1}-j2\Omega$	
—	$-18.8-j\,1623.1$	$\lambda^B_{-1(-1)}$	$\mu^B_{-1}-j2\Omega$	
$-26.2-j\,2543.8$	$-25.8-j\,2540.3$	$\lambda^B_{1(-1)}$	$\mu^B_1-j2\Omega$	

$^*N = 4$. Pure real eigenvalues are not listed in the table.

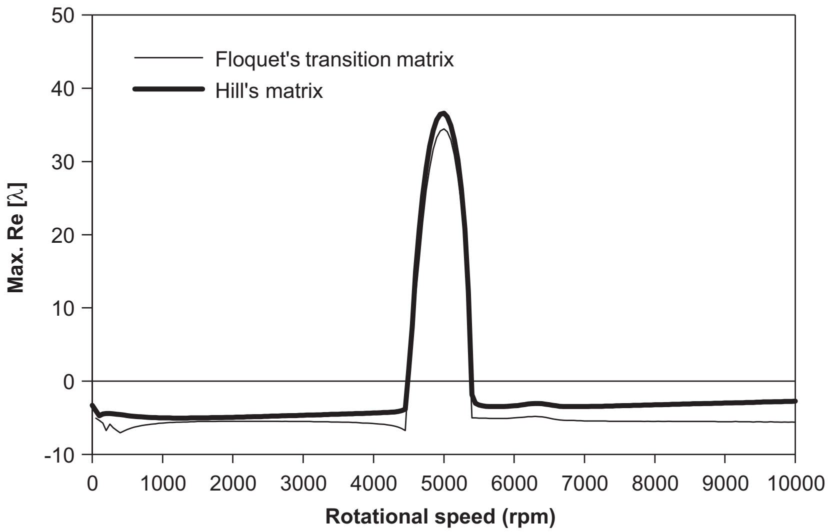

Fig. 3. Stability checks from Floquet's transition matrix and reduced Hill's matrix of order $6N$: $\delta = \Delta = 0.3$.

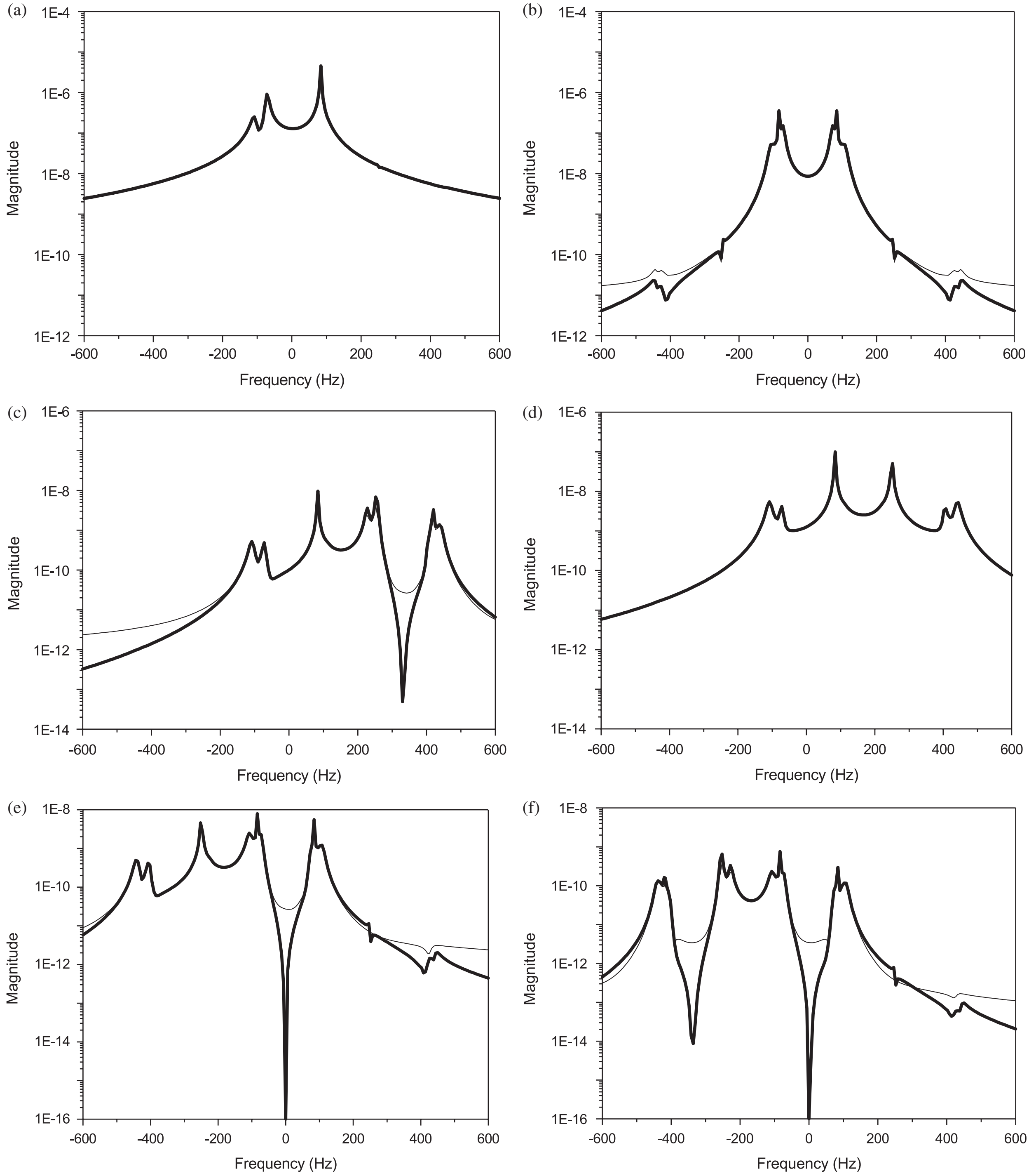

Fig. 4. Magnitude plots in m/N of dFRFs obtained from use of reduced Hill's matrix of order $6N$ (thick lines) and Floquet theory (thin lines): (a) $H_{g_{1;0}p_{1;0}}(\omega)$, (b) $H_{\bar{g}_{1;0}p_{1;0}}(\omega)$, (c) $H_{g_{1;-1}p_{1;0}}(\omega)$, (d) $H_{\bar{g}_{1;-1}p_{1;0}}(\omega)$, (e) $H_{g_{1;1}p_{1;0}}(\omega)$, and (f) $H_{\bar{g}_{1;1}p_{1;0}}(\omega)$: @ 10,000 rev/min (166.7 Hz); $\delta = \Delta = 0.3$.

of order $6N$ is found to yield fairly accurate results. The reduced Hill's matrix of order higher than $6N$ has been attempted, but the results remain almost unchanged, except appearance of additional new modes. The extra modes, which are the modulated modes of higher order and their complex conjugate modes, are associated with the high order coupling between the rotating and stationary asymmetry in the rotor system and they are not likely to play a significant role in the system response and stability in most of practical applications, unless both stationary and rotating asymmetries are very large. The Floquet approach, using the three term approximation (the index m was taken to be -1, 0 and 1 in Eq. (26)) for the complex Fourier-series expansion of the time-vaying eigenvectors, also results in the basic eigenvalues belonging to cluster 0, which are almost identical to the results in Table 1 obtained from the reduced Hill's matrix of order $6N$. But, the eigenvalues belonging to clusters -1 and 1 are automatically generated using relation (45a), and the corresponding complex Fourier coefficient vectors are chosen from cluster 0 according to relations (45b) and (45c).

The instability has also been checked using the maximum real part of eigenvalue obtained from the Floquet's transition matrix and the Hill's reduced order determinant, as shown in Fig. 3, respectively. Note that both results are in fairly good agreement with each other.

Fig. 4 compares the dFRFs calculated using the reduced order Hill's matrix of order $6N$ and the equivalent Floquet approach with three-term approximation at the rotational speed of 10,000 rev/min. The eigenvalues estimated from both methods are in good agreement, but the number of assumed modes used for calculation of dFRFs is kept unchanged for the former method, but it varies for the latter method, depending upon the type of dFRFs, due to the inherent nature of approximation. It often leads to relatively large discrepancies in the logarithmically scaled dFRF estimates obtained by two methods. The dFRFs shown in Fig. 4(a) and (d) are almost identical, but other types of dFRFs show some discrepancies with the order of magnitude less by 5–6 than the dominant peak values in the normal dFRFs, which corresponds perhaps to the measurement noise level in practice. In general, the number of assumed modes used in the Floquet method is less than, or equal at best to, the coordinate transform method. Thus, it can be concluded that the coordinate transform method is superior in estimation of dFRFs than the Floquet method. Note that the coordinate transform method, which is essentially a frequency domain approach, succeeds in approximating the dFRFs with a limited number of assumed modes (Ritz vectors), whereas the Floquet method, which is essentially a time domain approach, fails in using an effective set of base harmonics required to better estimate the dFRFs in the frequency domain. Theoretically speaking, as the number of assumed modes increases indefinitely, both methods will eventually lead to the identical results. Although there exist some discrepancies in the logarithmic magnitudes of dFRFs, the response calculations in the time domain by both methods yield little difference.

5. Conclusions

The complete complex modal analysis by two different approaches is developed for periodically time-varying linear rotor systems: one by employing Floquet theory and another by coordinate transformation. It is found that the coordinate transform method is not only straightforward in formulating the eigenvalue problem associated with constant system matrices of infinite order but also computationally efficient in calculating the eigensolutions and frequency response functions, whereas the Floquet method provides clear physical understanding of the eigenvalues and the corresponding eigenvectors.

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Appendix A. Construction of Hill's infinite order matrix by Floquet theory

Substituting the relations of modal vectors

$$\begin{Bmatrix} \mathbf{p}(t) \\ \bar{\mathbf{p}}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}(t) \\ \hat{\mathbf{u}}(t) \end{Bmatrix} e^{\mu t} = \sum_{m=-\infty}^{\infty} \begin{Bmatrix} \mathbf{u} \\ \hat{\mathbf{u}} \end{Bmatrix}_{(m)} e^{(j2m\Omega+\mu)t} \tag{A.1}$$

into the homogeneous part of Eq. (3), we obtain

$$\sum_{m=-\infty}^{\infty}\left[\mathbf{D}_{\mathbf{f};-m}\mathbf{u}_{(m)}+\mathbf{D}_{\mathbf{b};-m}\hat{\mathbf{u}}_{(m)}+\mathbf{D}_{\mathbf{r};-m}\hat{\mathbf{u}}_{(m)}\mathrm{e}^{\mathrm{j}2\Omega t}\right]\mathrm{e}^{(\mathrm{j}2m\Omega+\mu)t}=0,$$
$$\sum_{m=-\infty}^{\infty}\left[\bar{\mathbf{D}}_{\mathbf{f};m}\hat{\mathbf{u}}_{(m)}+\bar{\mathbf{D}}_{\mathbf{b};m}\mathbf{u}_{(m)}+\bar{\mathbf{D}}_{\mathbf{r};(m)}\mathbf{u}_{(m)}\mathrm{e}^{-\mathrm{j}2\Omega t}\right]\mathrm{e}^{(\mathrm{j}2m\Omega+\mu)t}=0 \tag{A.2}$$

or equivalently, the infinite set of algebraic equations given by

$$\mathbf{D}_{\mathbf{f};-m}\mathbf{u}_{(m)}+\mathbf{D}_{\mathbf{b};-m}\hat{\mathbf{u}}_{(m)}+\mathbf{D}_{\mathbf{r};-(m-1)}\hat{\mathbf{u}}_{(m-1)}=0,\quad \bar{\mathbf{D}}_{\mathbf{f};m}\hat{\mathbf{u}}_{(m)}+\bar{\mathbf{D}}_{\mathbf{b};m}\mathbf{u}_{(m)}+\mathbf{D}_{\mathbf{r};-(m-1)}\mathbf{u}_{(m-1)}=0, \tag{A.3}$$

where the matrices $\mathbf{D}_{\mathbf{f};m}(\mu+j2m\Omega)$, $\mathbf{D}_{\mathbf{b};m}(\mu+j2m\Omega)$ and $\mathbf{D}_{\mathbf{r};m}(\mu+j2m\Omega)$ take the form of Eq. (35c) with relation (45a). Similar relations can be derived with the adjoint vectors

$$\begin{Bmatrix}\mathbf{v}(t)\\ \hat{\mathbf{v}}(t)\end{Bmatrix}\mathrm{e}^{\mu t}=\sum_{m=-\infty}^{\infty}\begin{Bmatrix}\mathbf{v}\\ \hat{\mathbf{v}}\end{Bmatrix}_{(m)}\mathrm{e}^{(\mathrm{j}2m\Omega+\mu)t}.$$

In the above expressions, the sub- and superscripts used for mode identification are omitted for notational simplicity.

Eq. (A.3) can be rewritten as

$$\begin{bmatrix}\ddots & & & & & & \iddots\\ & \bar{\mathbf{D}}_{\mathbf{f};1} & \bar{\mathbf{D}}_{\mathbf{b};1} & & & & \\ & \mathbf{D}_{\mathbf{b};-1} & \mathbf{D}_{\mathbf{f};-1} & \mathbf{D}_{\mathbf{r};0} & & \mathbf{0} & \\ & & \bar{\mathbf{D}}_{\mathbf{r};1} & \bar{\mathbf{D}}_{\mathbf{f};0} & \bar{\mathbf{D}}_{\mathbf{b};0} & & \\ & & & \mathbf{D}_{\mathbf{b};0} & \mathbf{D}_{\mathbf{f};0} & \mathbf{D}_{\mathbf{r};1} & \\ & & \mathbf{0} & & \bar{\mathbf{D}}_{\mathbf{r};0} & \bar{\mathbf{D}}_{\mathbf{f};-1} & \bar{\mathbf{D}}_{\mathbf{b};-1}\\ & & & & & \mathbf{D}_{\mathbf{b};1} & \mathbf{D}_{\mathbf{f};1}\\ \iddots & & & & & & \ddots\end{bmatrix}\begin{Bmatrix}\vdots\\ \hat{\mathbf{u}}_{(1)}\\ \mathbf{u}_{(1)}\\ \hat{\mathbf{u}}_{(0)}\\ \mathbf{u}_{(0)}\\ \hat{\mathbf{u}}_{(-1)}\\ \mathbf{u}_{(-1)}\\ \vdots\end{Bmatrix}=\underset{\sim}{\mathbf{0}}, \tag{A.4}$$

which is referred to as the Hill's infinite order matrix equation with $3N$ bandwidth. Note that (A.4) becomes identical to Eq. (35a) with the relations (45), which is derived from the coordination transform approach. An alternative numerical method to improve the Floquet theory suggests using a reduced order Hill's matrix for calculation of the eigenvalues μ and the corresponding complex Fourier coefficient vectors, instead of erroneous integration of the Floquet's state transition matrix in the time domain. For example, the order of the reduced Hill's matrix becomes $6N$ for three term approximation, with $m=-1$, 0 and 1, in the complex infinite Fourier-series expansion of the periodically time-varying eigenvectors given in Eq. (A.1). And then, from the reduced Hill's matrix of order $6N$, we can obtain $12N$ sets of eigensolutions, consisting of the eigenvalues μ and the corresponding complex Fourier coefficient modal and adjoint vectors

$$\begin{Bmatrix}\mathbf{u}\\ \hat{\mathbf{u}}\end{Bmatrix}_{(m)}\ \text{and}\ \begin{Bmatrix}\mathbf{v}\\ \hat{\mathbf{v}}\end{Bmatrix}_{(m)}\quad \text{for}\ m=-1,0\ \text{and}\ 1.$$

Appendix B. A general rotor model

Consider the general rotor system, consisting of an asymmetric rotor with a rigid disk and two supporting bearings at ends of the mass-less shaft, as shown in Fig. 1. The orthotropic bearing stiffness and damping coefficients are assumed independent of the rotational speed. For analytical simplicity, the rotating (body-fixed) coordinates $\xi-\eta$ are assumed to be aligned with the principal mass moment of inertia axes of the disk and the principal shaft bending stiffness directions. Then, the equation of motion for the general rotor model reduces to Eq. (1) with the system matrices given by

$$\mathbf{p} = [p_1, p_2, p_3, p_4]^{\mathrm{T}} = [y_d + jz_d,\ \ \theta_y + j\theta_z,\ \ y_1 + jz_1,\ \ y_2 + jz_2]^{\mathrm{T}},$$

$$\mathbf{g} = [g_1, g_2, g_3, g_4]^{\mathrm{T}} = [f_{y_d} + jf_{z_d},\ \ f_{\theta y} + jf_{\theta z},\ \ 0,\ \ 0]^{\mathrm{T}},$$

$$\mathbf{M_f} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M_r} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Delta J & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}^d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & J_p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C_b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta c_{b1} & 0 \\ 0 & 0 & 0 & \Delta c_{b2} \end{bmatrix}, \quad \mathbf{K_b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta k_{b1} & 0 \\ 0 & 0 & 0 & \Delta k_{b2} \end{bmatrix},$$

$$\mathbf{C_f^d} = \begin{bmatrix} c_r & 0 & -l_2 c_r & -l_1 c_r \\ 0 & c_\theta & -jc_\theta/L & jc_\theta/L \\ -l_2 c_r & jc_\theta/L & l_2^2 c_r - c_\theta/L^2 & l_1 l_2 c_r - c_\theta/L^2 \\ -l_1 c_r & -jc_\theta/L & l_1 l_2 c_r - c_\theta/L^2 & l_1^2 c_r - c_\theta/L^2 \end{bmatrix}, \quad \mathbf{C_f^b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{b1} & 0 \\ 0 & 0 & 0 & c_{b2} \end{bmatrix},$$

$$\mathbf{C_f} = \mathbf{C_f^d} + \mathbf{C_f^b} - j\Omega\mathbf{G}^d, \quad \mathbf{C_r} = j2\Omega\mathbf{M_r}, \tag{B.1}$$

$$\mathbf{K_f^s} = \begin{bmatrix} k_r & -jk_{r\theta} & -l_2 k_r + k_{r\theta}/L & -l_1 k_r - k_{r\theta}/L \\ jk_{r\theta} & k_\theta & -jl_2 k_{r\theta} + jk_\theta/L & -jl_1 k_{r\theta} - jk_\theta/L \\ -l_2 k_r + k_{r\theta}/L & jl_2 k_{r\theta} - jk_\theta/L & l_2^2 k_r - 2l_2 k_{r\theta}/L + k_\theta/L^2 + k_{b1} & l_1 l_2 k_r + (2l_2 - 1)k_{r\theta}/L - k_\theta/L^2 \\ -l_1 k_r - k_{r\theta}/L & jl_1 k_{r\theta} + jk_\theta/L & l_1 l_2 k_r + (2l_2 - 1)k_{r\theta}/L - k_\theta/L^2 & l_1^2 k_r + 2l_1 k_{r\theta}/L + k_\theta/L^2 + k_{b2} \end{bmatrix},$$

$$\mathbf{K_r} = \begin{bmatrix} \Delta k_r & j\Delta k_{r\theta} & -l_2\Delta k_r + \Delta k_{r\theta}/L & -l_1\Delta k_r - \Delta k_{r\theta}/L \\ j\Delta k_{r\theta} & \Delta k_\theta & -jl_2\Delta k_{r\theta} - j\Delta k_\theta/L & -jl_1\Delta k_{r\theta} + j\Delta k_\theta/L \\ -l_2\Delta k_r + \Delta k_{r\theta}/L & -jl_2\Delta k_{r\theta} - j\Delta k_\theta/L & l_2^2\Delta k_r - 2l_2\Delta k_{r\theta}/L - \Delta k_\theta/L^2 & l_1 l_2\Delta k_r + (2l_2 - 1)\Delta k_{r\theta}/L + \Delta k_\theta/L^2 \\ -l_1\Delta k_r - \Delta k_{r\theta}/L & -jl_1\Delta k_{r\theta} + j\Delta k_\theta/L & l_1 l_2\Delta k_r + (2l_2 - 1)\Delta k_{r\theta}/L + \Delta k_\theta/L^2 & l_1^2\Delta k_r + 2l_1\Delta k_{r\theta}/L - \Delta k_\theta/L^2 \end{bmatrix},$$

$$\mathbf{K_f} = \mathbf{K_f^s} - j\Omega\mathbf{C_f^d}, \quad L = L_1 + L_2, \quad l_1 = \frac{L_1}{L}, \quad l_2 = \frac{L_2}{L},$$

where the physical parameters are

$$\begin{aligned} &J = (J_\xi + J_\eta)/2, \quad \Delta J = (J_\xi - J_\eta)/2, \quad c_r = (c_\xi + c_\eta)/2, \quad \Delta c_r = (c_\xi - c_\eta)/2, \\ &c_\theta = (c_{\theta_\xi} + c_{\theta_\eta})/2, \quad \Delta c_\theta = (c_{\theta_\xi} - c_{\theta_\eta})/2, \quad k_r = (k_\xi + k_\eta)/2, \quad \Delta k_r = (k_\xi - k_\eta)/2, \\ &k_\theta = (k_{\theta_\xi} + k_{\theta_\eta})/2, \quad \Delta k_\theta = (k_{\theta_\xi} - k_{\theta_\eta})/2, \quad c_b = (c_y + c_z)/2, \quad \Delta c_b = (c_y - c_z)/2, \\ &k_{r\theta} = (k_{\xi\theta_\eta} + k_{\eta\theta_\xi})/2, \quad \Delta k_{r\theta} = (k_{\xi\theta_\eta} - k_{\eta\theta_\xi})/2, \quad k_b = (k_y + k_z)/2, \quad \Delta k_b = (k_y - k_z)/2. \end{aligned} \tag{B.2}$$

Here, J_p and J are the polar and diametrical mass moments of inertia of the disk; c_r and c_θ (k_r and k_θ) are the shaft linear and angular internal dampings (stiffnesses); kr_θ is the coupled linear and angular stiffness of the shaft; c_b and k_b are the bearing damping and stiffness; y_d and z_d (θ_y and θ_z) denote the linear (angular) displacements of the disk; y_1 and z_1 (y_2 and z_1) denote the linear displacements of bearing #1 (#2), in the y–z directions; f_{y_d} and f_{z_d} ($f_{\theta y}$ and $f_{\theta z}$) are the forces (moments) acting on the disk in the y–z directions;

Δ indicates the perturbation. The shaft stiffnesses can be obtained from the structural mechanics as [15]

$$\begin{aligned} &k_{\xi} = 3\left(EI_{\xi_1}/L_1^3 + EI_{\xi_2}/L_2^3\right), \quad k_{\eta} = 3E\left(I_{\eta_1}/L_1^3 + I_{\eta_2}/L_2^3\right), \quad k_{\xi\theta_\eta} = 3E\left(-I_{\xi_1}/L_1^2 + I_{\xi_2}/L_2^2\right), \\ &k_{\eta\theta_\xi} = 3E\left(-I_{\eta_1}/L_1^2 + I_{\eta_2}/L_2^2\right), \quad k_{\theta_\xi\theta_\xi} = 3E\left(I_{\xi_1}/L_1 + I_{\xi_2}/L_2\right), \quad k_{\theta_\eta\theta_\eta} = 3E\left(I_{\eta_1}/L_1 + I_{\eta_2}/L_2\right), \end{aligned} \tag{B.3}$$

where E is the modulus of elasticity of the shaft, and, $I_{\xi_{1,2}}$ and $I_{\eta_{1,2}}$ are the area moments of inertia of shaft 1 and 2, with respect to the ξ and η axes, respectively.

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